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# CALCULATION OF LOAD DISTRIBUTION IN STIFFENED CYLINDRICAL SHELLS

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# CALCULATION OF LOAD DISTRIBUTION IN STIFFENED CYLINDRICAL SHELLS*

By H. Ebner and H. Köller

Thin-walled shells with strong longitudinal and transverse stiffening (for example, stressed-skin fuselages and wings) may, under certain simplifying assumptions, be treated as static systems with finite redundancies. In this report the underlying basis for this method of treatment of the problem is presented and a computation procedure for stiffened cylindrical shells with curved sheet panels indicated. A detailed discussion of the force distribution due to applied concentrated forces is given, and the discussion illustrated by numerical examples which refer to an experimentally determined circular cylindrical shell. The test results reported by E. Schapitz and G. Krümling in a simultaneously appearing paper (reference 1), confirm the reliability of the method proposed.

## I. INTRODUCTION

The determination of the stress distribution in stressed-skin (or monocoque) wings and fuselages, according to the elementary Navier theory for the bending of beams, or the Bredt theory for the torsion of thin-walled hollow bodies, is based on the preliminary assumption that the applied forces correspond to the elementary stress distribution. Large disturbances, which as characteristic additional stresses are superimposed on the elementary stresses, arise on the application of concentrated longitudinal forces - such, for example, as occur at a closed shell where the continuity of the structure is interrupted by a cutaway portion. (See fig. 1a.)

Further deviations from the simple state of stress indicated by the above theories arise on torsion and bending

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by transverse forces if the loads are applied at the intermediate transverse walls which are restrained from deformation, or if the dimensions are not uniform along the axis, or if the shell is rigidly attached at the end (fig. 1b). Finally, a small transverse stiffness of the bulkheads (as, for example, in monocoque fuselages) may, on the application of arbitrary transverse forces and moments, affect the elementary stress distribution.

The computation of the additional stresses due to restraint against deformation of a transverse section in torsion, has already been treated by several authors (reference 2). A method of determining the additional stresses arising on the application of longitudinal forces in long, thin-walled, cylindrical shells has been given in a paper by H. Wagner and H. Simon (reference 3). For shells with strong longitudinal and transverse stiffening, a simplified shell model with redundant axial groups of forces has been proposed in a previous work, in which the force distribution is determined by a statically indeterminate computation.[1] This method has been applied to the treatment of the special case of the 8-stringer cylindrical shell with double cross-sectional symmetry under axial load (reference 5). In the present paper the underlying principles are presented in a comprehensive manner and the computation is extended to cylindrical shells of arbitrary cross section under bending, twisting, and axial loading. For the latter case there is investigated, with the aid of numerical examples, the effect of various stiffeners on the force distribution. The method is further applied to a cylindrical shell whose stress distribution has been experimentally determined for the case of axial loading. (See reference 1.) A corresponding computation for the case of stiffened flat disks will appear in a later work in connection with similar test results.

## II. UNDERLYING PRINCIPLES OF THE METHOD

### 1. Simplified Model of the Shell Structure

The cylindrical shells consist essentially of the sheet or skin, the longitudinal stiffeners (stringers),

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[1] The same simplifications for the computation of stiffened shells are given by O. S. Heck in his paper (reference 4). His procedure, illustrated by examples, differs essentially from the one here presented in the choice of the static redundancies.

and the transverse stiffeners (bulkheads). The longitudinal stiffeners are assumed to be inside and attached to the skin, while the transverse stiffeners, assumed as parallel, may either be attached to the sheet (for example, in the neighborhood of the positions of load application) or attached only to the inner sides of the stringers. The bulkheads may be in the form of frameworks or flexurally rigid solid rings. The shell cross section is assumed to be simply connected, although the method may be extended to multiply connected systems. For the sake of simplicity, we shall restrict ourselves to cylindrical shells since the disturbances of the elementary stress condition in monocoque fuselages and wings almost always extend over individual small portions, which may approximately be considered as of cylindrical shape.

The transverse stiffening walls are denoted in the positive $x$ direction by $0, 1, \ldots, k, \ldots, n$ (fig. 2). They divide the system into bays, each of which is denoted by the stiffener number on its right. The stringers are numbered going around the positive direction from some initial position $1, 2, \ldots, j, \ldots, m$, the peripheral portions lying between the stiffeners (sheet or bulkhead portions) being denoted by the higher stringer number as subscript. All geometrical and structural magnitudes in the shell receive a double subscript $j,k$ corresponding to their position. If the transverse section possesses axes of symmetry, then symmetrically lying magnitudes are given the same subscripts.

For the computation of the shells under consideration, the following simplified model is used as a basis. The bulkheads take up forces that lie in the bulkhead planes only. The stringers are assumed to be hinge-connected to the bulkheads. In the case of stringers that go through and possess great flexural stiffness of their own, the additional stresses may be taken into account, as in the case of frameworks, by the introduction of nodal-point moments as additional static redundancies. In the skin itself a pure membrane-type of stress distribution is assumed.

The most important simplification is in connection with the stress distribution in the skin and stiffeners. The transverse stiffeners are, for the moment, assumed to be attached to the skin; a generalization will be considered under section III, 4. Normal stresses will then be transmitted by the stiffeners only, so that the sheet panels act as pure tension fields. The normal stiffness of the skin is taken into account by the addition of an effective width to the stringer sections. Under these assumptions, in each of the sheet panels there is a constant "shear flow"[2] ($t = s\ \tau$) to which correspond linearly distributed normal forces in the longitudinal stiffeners within the panels, while the transverse stiffener walls are stressed by tangential peripheral loads, namely, differences of the shear flows in the longitudinal direction. This state of stress agrees with the actual state - the more closely the stronger the stiffeners as compared with the skin. In the neighborhood of points of attachment and cut-out portions, these are the relations which generally exist. The reliability of the assumptions made, which had previously been applied to the computation of statically determinate systems (bending beams) is confirmed by tests. The assumption of constancy of the shear in each of the panels in the longitudinal direction may, in the case of curved shells, also be justified by the fact that generally the flexural stiffness of the skin and stringers is small compared to that of the transverse stiffness of the bulkheads, so that within no panel can important peripheral stresses arise in the skin.

This simplified shell model corresponds to a latticework with redundant transverse frames in which the diagonals are replaced by sheet panels under shear. For $(n - 1)$ intermediate transverse frames, each of which is q-fold statically indeterminate, the entire system with $n$ stringers is $[(n - 1)\ (n - 3) + (n + 1)q]$-fold statically indeterminate, and there are further additional redundancies at points of attachment. The statically indeterminate computation of the simplified system gives the force distribution "in the large" and the stress distribution in the actual shell design may then be estimated, taking into account the constructional details. (See the numerical example in section VI, 2.)

### 2. Principal Panel System
### Longitudinal Forces as Static Redundancies

The computation of the statically indeterminate model shell proceeds along lines similar to those developed by the first author in previous papers on space frameworks

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[2]The term "Schubflüsse," translated as "shear flow," is used consistently in German papers to denote shear stress times skin thickness. (Translator)

(reference 6) and box beams (reference 7). A suitable principal system is obtained if each intermediate bulkhead, in the case of end restraint also at the end bulkhead, (m - 3) longitudinal attachments of the m stringers are loosened by the insertion, for example, of hinges. On account of the static redundancies in the transverse frames, this main system is still statically indeterminate although only stresses of the same transverse stiffeners may be superimposed.

The redundant longitudinal forces in the resolved (m - 3) attachments of the stringers at a transverse stiffener produce in the two neighboring bays (m - 3) stress distributions independent of each other. Such independent stress conditions may also be obtained in a "principal panel system" in the following manner.[3] By inserting similar transverse stiffeners the bays are completely separated and between them there is introduced at the stringers (m - 3) systems of longitudinal forces under equilibrium - systems which we shall denote briefly by the term "characteristic force groups" (Eigenkraftgruppen) (fig. 3). These systems then produce in the neighboring bays (m - 3) independent stress distributions if none of the (m - 3) force groups may be represented by a linear combination of the remaining groups. We then say that such groups are linearly independent. At each transverse wall there are (m - 3) linearly independent characteristic force groups - for example, those consisting of a concentrated force and its "reaction forces" at three definite stringers not lying in a plane. The linear independence of (m - 3) arbitrarily set-up characteristic force groups may be established by the following criterion: There is formed the determinant of the (m - 3)d order in which each row contains the forces composing a group at (m - 3) stringers where the remaining three stringers do not lie in a plane. The columns then contain the individual forces of the various force groups at some definite stringer. The (m - 3) characteristic force groups will then be linearly independent if - and only if - this determinant is different from zero.

__Example:__ Figure 3 shows three characteristic force groups for a 6-stringer shell. The determinant, for example, of the individual forces at the three upper stringers will be

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[3]See detailed presentation cited under references 5 and 6.

$$\begin{vmatrix} -1 & 0 & +1 \\ +1 & -2 & +1 \\ +\frac{y_0}{2y_1} & -1 & +\frac{y_0}{2y_1} \end{vmatrix} = 2\left(\frac{y_0}{y_1} - 1\right)$$

and on account of $\frac{y_0}{y_1} \neq 1$ is different from zero, so that the force groups are linearly independent.

Such linearly independent force groups at the $k^{th}$ transverse stiffener and denoted by $X_{1,k}$, $X_{2,k}$, ... $X_{\mu,k}$, ..., $X_{m-3,k}$, are introduced as static redundancies and chosen as uniform in the axial direction - i.e., the component forces $P_j^{(\mu)}$ of the unit state of $X_{\mu,k}$ are the same for all k's. In this representation of a principal system, it is to be noted that the double transverse walls are identical, so that the stress states in the right bulkhead of the $k^{th}$ bay and in the left bulkhead of the $(k+1)^{th}$ bay must be superimposed. In the system so chosen the stress states of the redundancies at the $k^{th}$ transverse stiffener are superimposed only by those at the $(k-1)^{th}$ and $(k+1)^{th}$ transverse stiffeners at the $k^{th}$ and $(k+1)^{th}$ bays, respectively, and by those at the $(k-2)^{th}$ and $(k+2)^{th}$ transverse stiffener at the $(k-1)^{th}$ and $(k+1)^{th}$ bays. The redundancies at all other bulkheads do not directly affect the redundancies at the $k^{th}$ bulkhead.

From an arbitrary system of $(m - 3)$ linearly independent force groups $X_\mu$, it is possible to pass by a transformation

$$\overline{X}_\mu = c_{\mu,1} X_1 + c_{\mu,2} X_2 + \ldots + c_{\mu,m-3} X_{m-3}$$

$$(\mu = 1, 2, \ldots, m - 3) \tag{1}$$

to new linearly independent force groups $\overline{X}_\mu$, where the determinant formed from the coefficients $c_{\mu,\nu}$ must be different from zero. This arbitrariness in the choice of the static redundancies may be utilized to obtain considerable simplification in the computation. For arbitrary force groups the statically indeterminate computation leads, in the case of $(n - 1)$ intermediate transverse walls, to a system of $(n - 1)(m - 3)$ and $n(m - 3)$ elasticity equations for free and restrained end rings, respectively - the solution of which equations becoming very tedious when $n$ and $m$ are large. An attempt is therefore made to choose the $(m - 3)$ force groups in such a manner that only the uniform groups in the axial direction affect each other. The elasticity equations then break up into $(m - 3)$ independent partial systems, each with $(n - 1)$ and $n$ equations, respectively, which with the principal system chosen, are composed of five members each, and in the case of transverse stiffeners rigid in their planes, are composed of three members.

The introduction of such "orthogonal"[4] characteristic force groups is, in the case of an arbitrary number of stringers, possible only for a special shell form, namely, where there is cyclical symmetry with regard to the geometry and the elastic properties of the shell. The component forces $P_j^{(\mu)}$ of each of the groups are then to be set proportional to the ordinates of the sine and cosine curves with 2, 3, ... waves over the shell circumference.[5] For $m = 12$ these force groups are shown in figure 4. (The circular functions with higher wave numbers lead to the same groups.) With the aid of the orthogonality relations of the circular functions, it may easily be shown that the groups of forces thus formed constitute systems in equilibrium, and that in their mixed displacement coefficients (see section III, 2) the individual contributions from stringers, sheet and transverse stiffeners vanish. They are therefore orthogonal, as may be seen more simply from a consideration of the properties of symmetry, since two force groups - of which one is symmetrical with respect to an axis of symmetry of the system and the other antisymmetrical - do not influence one another.

[4]The mutual effect of two such force groups is measured by the so-called "mixed displacement coefficient" (see III, 2), which is essentially a sum or integration of products of the corresponding stress values. The vanishing of such products is spoken of in mathematics as expressing the "orthogonality," for example, of the vectors or functions concerned.

[5]This procedure corresponds to the deformation method proposed by Southwell for the computation of cyclically symmetrical space frameworks (reference 8).

The redundant force groups may be so ordered that with increasing subscript $\mu$ more individual groups which are in equilibrium over a smaller and smaller region of the circumference may be formed from their component forces; for example, in figure 4, for the sine and cosine groups for $\mu = 2$, equilibrium is possible over the entire circumference only, whereas the groups for $\mu = 3$ may each be split up into two subgroups which may be in equilibrium over only half the circumference. According to the principle of St. Venant, the regions of influence of the groups must decrease with increasing "order" of subscript $\mu$, so that after a certain order they may be neglected.

In the case of a shell whose structure deviates only slightly from that corresponding to cyclical symmetry, force groups are formed corresponding to complete cyclical symmetry and the slight mutual effects are neglected. For shells without cyclical symmetry, with only a few stringers and with sufficient axial symmetry of cross section, it is still possible to set up orthogonal force groups. An example is that shown in figure 3 for a 6-stringer shell with cross-sectional symmetry with respect to the $y$ and $z$ axes. Also for 12-stringer, doubly symmetrical shells, characteristic force groups that have a negligible effect on each other may, as shown in section V, be set up for the most important loading conditions.

It is shown in the Appendix that for an arbitrary shell construction, complete orthogonality of the force groups can be attained in general only if the effect of the deformations of one of the three shell-structure elements is neglected - if, for example, rigid sheet or rigid transverse stiffener is assumed. The force groups may in this case be set up, under the assumption of equal panels, with the aid of "principal axes transformation." This procedure, however, possesses only theoretical value. For most practical cases it is possible to use the simple approximate system considered in section V. An alternative is to restrict the computation to a few panels in the neighborhood of points of disturbance and solve the elasticity equations for arbitrary force groups according to the iteration method described in section III, 3.

## 3. Decomposition of External Loading Principal Substitute System

We consider the loading cases in which there is a considerable disturbance from the elementary state of stress. This will be the case where concentrated forces are applied and also the case of twisting and transverse bending of restrained shells or shells with "stepped dimensions," and also the case where load is applied at the intermediate bulkheads.

The loading of the shell by concentrated axial forces is generally reduced to the application of characteristic force groups. For this purpose there is subtracted from the external loading the forces corresponding to the elementary theory and there is thus obtained at the transverse stiffener at which the load is applied, a force group which gives rise in the shell to a state of stress expressing the difference between the elementary and actual stress. Each force group $X_o$ acting at a transverse stiffener may, however, be split up into groups of the form of the introduced redundancies $X_1, X_2, \ldots, X_{n-3}$

$$X_0 = c_1 X_1 + c_2 X_2 + \ldots + c_{n-3} X_{n-3} \qquad (2)$$

since for the component forces of the groups at $(m - 3)$ stringers, this equation represents a system of $(m - 3)$ linear equations with the $(m - 3)$ values $c_1, c_2, \ldots, c_{n-3}$ as unknowns. This system has a unique solution since the determinant formed out of the $(m - 3)$ component forces of the linearly independent redundancies, according to the criterion of section II, 2, is different from zero. The manner of this decomposition is indicated in figure 5 for the case of a bending force group composed of four concentrated forces at a 12-stringer shell. The characteristic force groups $Y_1$ and $Y_2$ correspond to the cosine groups for $\mu = 3$ and $\mu = 5$ in figure 4, and are considered more in detail in section V.

For the external loading, or a portion of it, another principal system than that used for the static redundancies may be used in a statically indeterminate computation. This "substitute principal system" is chosen so that its state of stress corresponds as closely as possible to the final one - the statically indeterminate computation then constituting only an added computation; or a substitute principal system, for which the computation of the load coefficients is as simple as possible, is taken. Both points of view, however, may not in general, be satisfied simultaneously. In applying a bending force group consisting of four concentrated forces (fig. 5) on a restrained shell, the four stringers at which the load is applied may, for example, serve as the substitute principal system and the load coefficients are then determined only by integration over the normal stress distributions in these stringers. The static redundancies, however, then assume large values over the entire shell length. If, however, the bending force group is reduced, as described, to the application of characteristic force groups, the static redundancies die down to small values in the longitudinal direction. When the shell is loaded by twisting forces, the "Bredt tube," and by transverse bending forces, the "ideal beam," respectively, are chosen as the substitute principal systems. The statically indeterminate computation then gives the deviations due to the restraint against deformation, from the elementary stress condition.

There occur, besides, under these loading conditions, disturbances due to the application at the transverse stiffeners, of forces not corresponding to the elementary theories. By subtracting the equivalent elementary stress distribution in the transverse stiffener from that actually applied, there is obtained an equilibrium system of external forces which deform the transverse stiffeners and thus affect the axial force distribution. As a rule, however, the statically indeterminate computation can be dispensed with if the bulkheads at which the loads are applied are designed sufficiently stiff.

## III. PROCEDURE FOR THE STATICALLY INDETERMINATE COMPUTATION

### 1. Stress Distribution

The following notation is used for the dimensions of the cylindrical shell under consideration (fig. 2):

$a_k$, length of $k^{th}$ bay (bulkhead spacing)

$b_j$, width of sheet panel between $(j - 1)^{th}$ and $j^{th}$ stringer.

$U = \sum_{j=1}^{m} b_j$, circumference of shell.

$F = \sum_{j=1}^{m} F_j$, cross-sectional area enclosed by shell.

$F_{j,k}$, cross section of the $j^{th}$ stringer in the $k^{th}$ bay (stringer section plus effective skin section).

$s_{j,k}$, skin thickness in $k^{th}$ panel between $(j-1)^{th}$ and $j^{th}$ stringer.

For flexurally stiff ring bulkheads with constant cross section along the circumference:

$F_{S,k}$, cross section of $k^{th}$ bulkhead.

$J_{S,k}$, moment of inertia of $k^{th}$ bulkhead ring (with respect to radial deflection).

The component forces $P_j^{(\mu)}$ at the $j^{th}$ stringer for the unit state of the redundancies $X_{\mu,k}$ are equal for all values of $k$. The remaining symbols are sufficiently clear from the text and illustrations.

As a result of $X_{\mu,k} = 1$, the $k^{th}$ and $(k+1)^{th}$ panels in the principal system are stressed. According to the simplified model shell, normal forces arise in the stringers that decrease linearly to zero (fig. 6):

$$L_{j,k}^{(\mu)} = \frac{x}{a_k} P_j^{(\mu)}; \quad L_{j,k+1}^{(\mu)} = \frac{a_{k+1} - x}{a_{k+1}} P_j^{(\mu)} \tag{3}$$

(In each panel $x$ is measured from the left bulkhead) and from

$$\frac{\partial L_{j,k}}{\partial x} = -(t_{j+1,k} - t_{j,k})^{6}$$

there is obtained for the constant shear flow in the sheet fields

[6] The shear times thickness (shear flow) is chosen positive in the direction of positive shear stress, hence opposite that in the work cited in reference 2. In the figures the reactions of the shear flows on the stiffeners are shown.

$$t_{j,k}^{(\mu)} = t_{1,k}^{(\mu)} - \frac{1}{a_k} \sum_{i=1}^{j-1} P_i^{(\mu)}$$

$$t_{j,k+1}^{(\mu)} = t_{1,k+1}^{(\mu)} + \frac{1}{a_{k+1}} \sum_{i=1}^{j-1} P_i^{(\mu)} \qquad (4)$$

In particular cases $t_{1,k}^{(\mu)}$ and $t_{1,k+1}^{(\mu)}$ may be determined from considerations of symmetry of the system and of the force groups; in general, however, there is required the additional condition that the shear flows at the transverse stiffeners form a system in equilibrium whose moment therefore vanishes:

$$\sum_{j=1}^{m} t_{j,k}^{(\mu)} (2\underline{F}_j) = 0 \quad \text{and} \quad \sum_{j=1}^{m} t_{j,k+1}^{(\mu)} (2\underline{F}_j) = 0 \qquad (5)$$

where $\underline{F}_j$ is the sector of the cross-sectional area from a suitable origin (for example, center of gravity of shell cross section) to the circumferential portion $b_j$.

Having obtained the shear flows, the stress distribution in the transverse stiffener walls may be determined by considering the reactions of the shear flows as tangential peripheral loads, the stresses in the $(k - 1)^{th}$ and $(k + 1)^{th}$ bulkheads being opposite to the stress of the $k^{th}$ bulkhead from the shears of the $k^{th}$ and $(k + 1)^{th}$ panels, respectively. (See fig. 6.) The stress distribution depends on the design of the transverse stiffener and, with static indeterminancy, required an additional statically indeterminate computation. Framework and solid connections are generally stiff enough, so that the changes in shape in their planes are small compared to the deformations of the stringers and sheet. The axial force distribution will then be only slightly affected by them, so that for the statically indeterminate computation they may be assumed as rigid. The flexurally rigid bulkhead rings in monocoque fuselages, however, are subject to greater deformations which in any particular loading condition must be taken into account. In general, they are threefold statically indeterminate, although important simplifications result if axes of symmetry of the cross section are present. The bulkhead ring computation for several shell shapes is given in section V.

At the transverse stiffener 0, the applied characteristic force groups to which the axial force loading had been reduced (section II, 3) stress only the first panel of the panel system. The state of stress is correspondingly given by (3), (4), (5), where it is to be noted that the transverse stiffener 0 is loaded only by the shears of one panel.

In pure torsional loading by moments $M_\kappa$ at the $\kappa^{th}$ bulkhead, the $k^{th}$ panel is acted upon by a twisting moment:

$$T_k = \sum_{\kappa=0}^{k-1} M_\kappa \tag{6}$$

resulting, according to the Bredt theory, in a shear flow

$$t_k^{(0)} = - \frac{T_k}{2F} \tag{7}$$

in the skin constant over the circumference and within the panel. The stringers are free from stress, the bulkhead $k$ is loaded by the difference between the actual applied twisting moment and the uniformly distributed elementary moment.

For transverse force bending the determination of the "substitute state of stress" in the stringers and skin proceeds according to the well-known formulas of the beam theory under the assumption of the simplified model shell (reference 2). In bending about the z axis, which is passed through the center of gravity of the shell cross section, the axial forces in the $k^{th}$ panel are:

$$L_{j,k}^{(0)} = \frac{F_{j,k}\, y_j}{\sum_{i=1}^{m} F_{i,k}\, y_i^2} \bar{B}_z \tag{8}$$

and the shear flows:

$$t_{j,k}^{(0)} = t_{1,k}^{(0)} - \frac{\sum_{i=1}^{j-1} F_{i,k}\, y_i}{\sum_{i=1}^{m} F_{i,k}\, y_i^2} \bar{Q}_y \tag{9}$$

In the above equations $\overline{B}_z$ is the bending moment, $\overline{Q}_y$ the transverse force at the corresponding position, and $t_{i,k}^{(0)}$, when not obtained from the symmetry properties, is obtained in general from the condition:

$$\sum_{i=1}^{n} \frac{t_{i,k}^{(0)}\, b_i}{s_{i,k}\, G} = 0$$

for bending without torsion. In the transverse stiffener walls there arises, as in the case of torsion, a stress distribution due to the manner of load application not corresponding to the elementary theory.

## 2. Strain Distribution

From the stress distributions due to $X_\mu = 1$, $X_\nu = 1$, there are obtained in the usual way the strain or displacement coefficients $\delta_{\mu,\nu}$, that is, the virtual work which is done by the force group $X_\mu = 1$ against the displacements due to $X_\nu = 1$:

$$\delta_{\mu,\nu} = \sum_{\text{stringers}} \int \frac{L^{(\mu)}\, L^{(\nu)}}{E\, F}\, dx + \sum_{\text{skin}} s \iint \frac{t^{(\mu)}\, t^{(\nu)}}{s^2\, G}\, du\, dx$$

$$+ \sum_{\text{bulkheads}} \left[ \oint \frac{B^{(\mu)}\, B^{(\nu)}}{E\, J_S}\, du_s + \oint \frac{N^{(\mu)}\, N^{(\nu)}}{E\, F_S}\, du_s \right] \quad (10)$$

(du = circumferential element of skin,

$du_S$ = circumferential element of neutral ring axis.)

The bulkheads are assumed to be flexurally stiff rings in which, as in the case of the beam, the portion of the work done by the transverse forces, is neglected. For framework and solid connections, corresponding values are to be formed, the derivation of which will not be gone into since the changes in shape of such transverse walls are negligible for the static computation.

We consider first the individual contributions to the virtual work of the stringers, sheet, and bulkheads separately. For a stringer with cross section $F$ and length $a$, the portion due to the normal forces $L^{(\mu)}$ and $L^{(\nu)}$ which vary linearly between the end values $P_o^{(\mu)}$, $P_o^{(\nu)}$ and $P_a^{(\mu)}$, $P_a^{(\nu)}$ is:

$$\frac{a}{6EF}\left[P_o^{(\mu)}\ (2P_o^{(\nu)} + P_a^{(\nu)}) + P_a^{(\mu)}\ (2P_a^{(\nu)} + P_o^{(\nu)})\right] \qquad (11)$$

The displacement coefficient $w_{L,k}^{(\mu,\nu)}$ of the stringers of the $k^{th}$ bay due to $X_{\mu,k} = 1$ and $X_{\nu,k} = 1$ (or $X_{\mu,k-1} = 1$ and $X_{\nu,k-1} = 1$):

$$w_{L,k}^{(\mu,\nu)} = \frac{a_k}{3} \sum_{j=1}^{n} \frac{P_j^{(\mu)}\ P_j^{(\nu)}}{E\ F_{j,k}} \qquad (12)$$

A sheet of dimensions $a$, $b$ and shear stiffness $s\ G$ contributes the portion due to the constant shear flows $t^{(\mu)}$ and $t^{(\nu)}$:

$$\frac{ab}{s\ G}\ t^{(\mu)}\ t^{(\nu)} \qquad (13)$$

so that the displacement coefficient $w_{B,k}^{(\mu,\nu)}$ of the sheet portion of the $k^{th}$ bay due to $X_{\mu,k} = 1$ and $X_{\nu,k} = 1$ (or $X_{\mu,k-1} = 1$ and $X_{\nu,k-1} = 1$) is:

$$w_{B,k}^{(\mu,\nu)} = a_k \sum_{j=1}^{n} \frac{b_j}{s_{j,k}\ G}\ t_{j,k}^{(\mu)}\ t_{j,k}^{(\nu)} \qquad (14)$$

The displacement coefficient $w_{S,k}^{(\mu,\nu)}$ of the $k^{th}$ transverse stiffener of each redundancy $X_{\mu,k} = 1$ and $X_{\nu,k} = 1$ loaded only by the shears of the $k^{th}$ bay in the case of flexurally stiff rings with constant cross section along the circumference, amount to

$$w_{S,k}^{(\mu,\nu)} = \frac{U\ F^2}{4a_k^2\ E\ J_{S,k}}\ \kappa_{S,k}^{(\mu,\nu)} \qquad (15)$$

with

$$\kappa_{S,k}^{(\mu,\nu)} = \frac{4a_k^2}{U\,F^2}\left[\oint B_k^{(\mu)}\,B_k^{(\nu)}\,du_{S,k} + i_{S,k}^2 \oint N_k^{(\mu)}N_k^{(\nu)}\,du_{S,k}\right]$$

$$\left(I_{S,k} = \frac{J_{S,k}}{F_{S,k}}\right) \qquad (16)$$

where $B_k^{(\mu)}$, $B_k^{(\nu)}$ denote the bending moments $N_k^{(\mu)}$, $N_k^{(\nu)}$ the normal forces in the $k^{th}$ bulkhead ring produced by the shears in the $k^{th}$ bay due to $X_{\mu,k} = 1$ and $X_{\nu,k} = 1$. The coefficient $\kappa_{S,k}^{(\mu,\nu)}$ depends besides on the type and magnitude of the redundancies, only on the geometric shape of the $k^{th}$ bulkhead, the position of its neutral axis, and the stringer center of gravity. The coefficient varies little with the cross-sectional shape and is given in section V for several simple shell shapes. If the $k^{th}$ bulkhead is loaded only by the shears of the $k^{th}$ and the $(k + 1)^{th}$ bays or only by the shears of the $(k + 1)^{th}$ bay, the coefficients are, respectively:

$$\frac{a_k}{a_{k+1}}\,\kappa_{S,k}^{(\mu,\nu)} \quad \text{and} \quad \frac{a_k^2}{a_{k+1}^2}\,\kappa_{S,k}^{(\mu,\nu)}$$

since the shears in the two bays are proportional to $\frac{1}{a_k}$ and $\frac{1}{a_{k+1}}$.

From these separate portions the total displacement coefficients $\delta_{k,k}^{(\mu,\nu)}$ of the redundancies at the $k^{th}$ transverse stiffener may be expressed in the form:

$$\delta_{k,k}^{(\mu,\nu)} = \left[w_{L,k}^{(\mu,\nu)} + w_{L,k+1}^{(\mu,\nu)}\right] + \left[w_{B,k}^{(\mu,\nu)} + w_{B,k+1}^{(\mu,\nu)}\right]$$

$$+ \left[\left(\frac{a_{k-1}}{a_k}\right)^2 w_{S,k-1}^{(\mu,\nu)} + \left(1 + \frac{a_k}{a_{k+1}}\right)^2 w_{S,k}^{(\mu,\nu)} + w_{S,k+1}^{(\mu,\nu)}\right] \quad (17a)$$

In the longitudinal direction the redundancies $X_{\mu,k}$ at the $k^{th}$ bulkhead are combined with those of the

$(k-1)^{th}$ and $(k+1)^{th}$ bulkheads, respectively, ($X_{\nu,k-1}$ and $X_{\nu,k+1}$) through superposition of the stress distributions in the $k^{th}$ and $(k+1)^{th}$ bays to form the coefficients $\delta_{k,k-1}^{(\mu,\nu)}$ and $\delta_{k,k+1}^{(\mu,\nu)}$, respectively. Since the force groups at the different transverse stiffeners are chosen uniform there hold the symmetrical relations:

$$\delta_{k,k-1}^{(\mu,\nu)} = \delta_{k,k-1}^{(\nu,\mu)} = \delta_{k-1,k}^{(\mu,\nu)} = \delta_{k-1,k}^{(\nu,\mu)}$$

and those "mixed displacement coefficients" may be expressed in terms of the $w$ values:

$$\delta_{k,k-1}^{(\mu,\nu)} = \frac{1}{2}\, w_{L,k}^{(\mu,\nu)} - w_{B,k}^{(\mu,\nu)}$$

$$- \left[\frac{a_{k-1}}{a_k}\left(1 + \frac{a_{k-1}}{a_k}\right) w_{S,k-1}^{(\mu,\nu)} + \left(1 + \frac{a_k}{a_{k+1}}\right) w_{S,k}^{(\mu,\nu)}\right] \qquad (17b)$$

Further, by superposition of the stress distributions in the $(k-1)^{th}$ and $(k+1)^{th}$ transverse stiffener there arise the mixed displacement coefficients $\delta_{k,k-2}^{(\mu,\nu)}$ and $\delta_{k,k+2}^{(\mu,\nu)}$, respectively, for which corresponding symmetrical relations hold and which may be expressed in terms of the $w_S$ values:

$$\delta_{k,k-2}^{(\mu,\nu)} = \frac{a_{k-1}}{a_k}\, w_{S,k-1}^{(\mu,\nu)} \qquad (17c)$$

In a similar manner are to be determined the coefficients of the redundancies by superimposing their unit states upon the stresses due to the external loading in the bay or substitute principal system according to (10). By the loading of the shell at the end stiffener 0 with the characteristic force groups $X_{1,0}$, $X_{2,0}$, ..., $X_{m-3,0}$ of the form of static redundancies, only the redundancies at the first and second transverse stiffeners are affected. The load coefficients $\delta_{1,0}^{(\mu,\nu)}$, $_{2,0}^{(\mu,\nu)}$ of the redundancies $X_{\mu,1}$, $X_{\mu,2}$ due to a characteristic force group $X_{\nu,0} = 1$

are therefore:

$$\delta_{1,0}^{(\mu,\nu)} = \frac{1}{2} w_{L,1}^{(\mu,\nu)} - w_{B,1}^{(\mu,\nu)} \left[w_{S,0}^{(\mu,\nu)} + \left(1 + \frac{a_1}{a_2}\right) w_{S,1}^{(\mu,\nu)}\right]$$

$$\delta_{2,0}^{(\mu,\nu)} = \frac{a_1}{a_2} w_{S,1}^{(\mu,\nu)} \qquad (18)$$

where $w_{S,0}^{(\mu,\nu)}$ is to be formed from the stresses of the end bulkhead 0 by the shears of the first bay due to $X_{\mu,1} = 1$ and $X_{\nu,1} = 1$. For an arbitrary force group $X_0$ at the transverse wall 0, and which according to (2) is split up in the form:

$$X_0 = \sum_{\nu=1}^{m-3} c_\nu X_{\nu,0}$$

the load coefficients of the redundancies $X_{\mu,1}$ and $X_{\mu,2}$ amount to

$$\delta_{1,0}^{(\mu)} = \sum_{\nu=1}^{m-3} c_\nu \cdot \delta_{1,0}^{(\mu,\nu)}; \quad \delta_{2,0}^{(\mu)} = \sum_{\nu=1}^{m-3} c_\nu \cdot \delta_{2,0}^{(\mu,\nu)} \qquad (19)$$

In pure torsion by the moments $M_k$ at the bulkheads and according to the elementary theory applied distributively, there follows according to (7) and (13) for the load coefficients $\delta_{k,0}^{(\mu)}$ of the redundancies $X_{\mu,k}$:

$$\delta_{k,0}^{(\mu)} = -\frac{T_k}{2F} \sum_{j=1}^{m} \frac{a_k\, b_j}{s_{j,k}\, G}\, t_{j,k}^{(\mu)} - \frac{T_{k+1}}{2F} \sum_{j=1}^{m} \frac{a_{k+1}\, b_j}{s_{j,k+1}\, G}\, t_{j,k+1}^{(\mu)} \qquad (20)$$

If the torsional moments are not applied distributively, according to the elementary theory, there are still to be added the displacement coefficients due to the stresses in the transverse walls.

In the same manner in the case of bending by transverse forces, the load coefficients may, with the aid of formulas (11), (13), and (16), be built up from the individual contributions of stringers, sheet, and bulkheads.

## 3. Elasticity Equations and Solution in the General Case

After the computation of the virtual work coefficients the magnitudes of the redundancies are determined from the elasticity equations which express the conditions that the deformations of neighboring bays at the common bulkhead should agree. For arbitrary characteristic force groups as redundancies these equations have the form:

$$\sum_{\nu=1}^{m-3}\left(\delta_{k,k-2}^{(\mu,\nu)} X_{\nu,k-2} + \delta_{k,k-1}^{(\mu,\nu)} X_{\nu,k-1} + \delta_{k,k}^{(\mu,\nu)} X_{\nu,k} + \right.$$

$$\left. + \delta_{k,k+1}^{(\mu,\nu)} X_{\nu,k+1} + \delta_{k,k+2}^{(\mu,\nu)} X_{\nu,k+2}\right) + \delta_{k,0}^{(\mu)} = 0 \qquad (21)$$

$k = 1, 2, \ldots., n - 1$ and $n$; $\mu = 1, 2, \ldots. m - 3$

with the boundary values:

$X_{\mu,-1} = X_{\mu,o} = 0$ and $X_{\mu,n} = X_{\mu,n+1} = 0$;

and $X_{\mu,n+1} = X_{\mu,n+2} = 0$

for a freely deformable end stiffener and end stiffener restrained against deformation, respectively.

The system of equations therefore breaks up, as shown in figure 7, for $n = 6$ and $m = 5$, into partial systems of five members each and which are formed in the direction of the principal diagonal from the displacement coefficients of uniform longitudinal force groups, while the remaining partial systems are composed of the mixed displacement coefficients of nonuniform force groups. If the force groups are orthogonal to one another, these "mixed" partial systems vanish. The values of the uniform redundancies in the longitudinal direction are then obtained from independent 5-member partial systems which only contain uniform redundancies (principal equations). For the solution of these 5-member elasticity equations - which in the case of rigid transverse stiffener walls are 3-member equations - there have been developed in static structure computations a number of suitable methods, so that it is not necessary to go into the matter any further. (See among others, the works cited in reference 9.) For regular systems the solutions may be given in finite form as has been done in section IV, for equal panels.

If the redundant force groups are not orthogonal to one another, it is generally possible to obtain the condition that the mixed coefficients of nonuniform groups be small compared to the coefficients of the principal equations. For more accurate computations these approximate solutions are improved step by step by substituting the values in the mixed partial systems and thus obtaining modified load members, for which the principal equations are again solved. This iteration process is most conveniently carried through in "single steps" - i.e., the new approximate values of a group are used directly to improve the next group. The process then always converges for elasticity equations (reference 10).

After the determination of the redundancies, the final force distribution may immediately be obtained by superimposing upon the load condition in the principal system the corresponding multiples of the unit states of the redundancies.

If $P^{(o)}_{j,k}$ is the force due to the external loading in the principal system, the force $P_{j,k}$ in the $j^{th}$ stringer at the $k^{th}$ bulkhead is:

$$P_{j,k} = P^{(o)}_{j,k} + \sum_{\mu=1}^{n-3} P^{(\mu)}_{j} X_{\mu,k} \tag{22}$$

For the shear flow $t_{j,k}$ in the $j^{th}$ panel of the $k^{th}$ bay, there is obtained:

$$t_{j,k} = t^{(o)}_{j,k} + \sum_{\mu=1}^{n-3} t^{(\mu)}_{j,k} \left(X_{\mu,k} - X_{\mu,k-1}\right) \tag{23}$$

where $t^{(o)}_{j,k}$ is the value in the principal system. Correspondingly there is found the stress of the $k^{th}$ bulkhead from the redundancies $X_{\mu,k-1}$, $X_{\mu,k}$, and $X_{\mu,k+1}$. The bending moment, for example, in the $k^{th}$ bulkhead ring is:

$$B_k = B_k^{(o)} + \sum_{\mu=1}^{n-3} B_k^{(\mu)} \left[- X_{\mu,k-1} + \left(1 + \frac{a_k}{a_{k+1}}\right) X_{\mu,k} - \frac{a_k}{a_{k+1}} X_{\mu,k+1}\right] \tag{24}$$

where $B_k^{(o)}$ is the value in the principal system, $B_k^{(\mu)}$ is the value due to the shears of the $k^{th}$ bay as a result of $X_{\mu,k} = 1$.

### 4. Extension of the Shell Model

If the transverse stiffeners are not attached to the skin but only fastened to the inner edges of the stringers, as is often the case with monocoque fuselages, the shell model thus far considered cannot be applied without modification. A transfer of the shears in the form of tangential loads to the bulkheads is then possible to a limited extent only, on account of the weak attachment. In order to take this effect into account, the limiting condition is considered where no tangential forces at all can be transmitted from the sheet to the transverse stiffeners, so that there is a steady transition of the shear at the stiffeners. A state of equilibrium in the case of variable longitudinal stresses is possible only through the setting up in the skin of peripheral stresses $\sigma_u$, which correspond to the variable longitudinal shear stresses. In the case of curved sheet with small bending stiffness of its own, the radial components of those peripheral stresses must be transmitted to the stringers which, as a result of their flexural stiffness, retransmit these forces to the transverse stiffeners. For the sake of simplicity, let this radial load of the transverse stiffeners be assumed as continuously distributed over the periphery as is approximately the case for closely spaced stringers.

We consider first the loading of the $k^{th}$ bulkhead for arbitrary axial shear distribution (fig. 8a). For the peripheral force $p_u = s\,\sigma_u$, there is obtained from the equilibrium equation $\frac{\partial p_u}{\partial u} + \frac{\partial t}{\partial x} = 0$:

$$p_u = -\int_{u_o}^{u} \frac{\partial t}{\partial x}\, du$$

where $u_o$ is a position at which the peripheral stress vanishes. The radial component $q$ per unit of area at a position with radius of curvature $r$ is then (fig. 8b):

$$q = -\frac{1}{r} p_u = \frac{1}{r}\int_{u_o}^{u} \frac{\partial t}{\partial x}\, du$$

At the transverse stiffeners themselves hinge connection is assumed at the stringers. The radial force R du over an element du which acts on the transverse stiffener as a "reaction force" may then be determined from the moment conditions. For the $k^{th}$ bulkhead, as a result of the shear in the $k^{th}$ bay (fig. 8c, $x$ is computed from the left bulkhead in each bay), there is obtained:

$$R_{k,k} = \frac{1}{a_k}\int_0^{a_k} x\, q\, dx = \frac{1}{a_k r}\int_0^{a_k} x\left[\int_{u_o}^{u} \frac{\partial t}{\partial x}\, du\right] dx$$

Interchanging the order of integration and integrating by parts with respect to $x$, there is obtained:

$$R_{k,k} = \frac{1}{a_k r}\int_{u_o}^{u}\left[a_k t_{(k)} - \int_0^{a_k} t\, dx\right] du = \frac{1}{r}\int_{u_o}^{u} [t_{(k)} - t_k^*]\, du \tag{25}$$

where $t_{(k)}$ is the shear flow at the $k^{th}$ bulkhead and

$$t_k^* = \frac{1}{a_k}\int_0^{a_k} t\, dx$$

is the mean value of the shear flow in the $k^{th}$ bay. The $k^{th}$ bulkhead is further loaded by the shears of the $(k + 1)^{th}$ bay. For the radial force $R_{k,k+1}$ at position $u$, there is obtained according to a computation similar to the above:

$$R_{k,k+1} = \frac{1}{a_{k+1}\, r}\int_{u_o}^{u}\left[\int_0^{a_{k+1}} (a_{k+1} - x)\frac{\partial t}{\partial x}\, dx\right] du =$$

$$\frac{1}{r}\int_{u_o}^{u} [- t_{(k)} + t_{k+1}^*]\, du \tag{26}$$

The total radial load of the $k^{th}$ bulkhead per unit circumferential distance is therefore:

$$R_k = R_{k,k} + R_{k,k+1} = \frac{1}{r} \int_{u_o}^{u} [t^*_{k+1} - t_k^*] \, du \qquad (27)$$

If no tangential loads can be transmitted, the radial load therefore depends only on the mean integrated values of the shears in the neighboring bays.

To compute the system, the "simple shear field scheme" according to section II, 1, is used as a starting basis. The longitudinal stiffnesses of skin and stringers are combined and in place of the variable shear in the longitudinal direction, there is taken the mean integral value within each bay and the corresponding linear force distribution in the stringers. In the statically indeterminate computation for the redundant axial force groups, there are neglected at each transverse stiffener wall the displacement contributions of the skin due to the peripheral stresses, and their effect is taken into account only through the varied loading of the transverse stiffeners with radial forces according to (27). There are then obtained other displacement coefficients $w_{S,k}^{(\mu,\nu)}$ than for the case of tangential loading. The difference depends, however, as will be shown by an example in section V, 1, only on the excentric position of skin bulkheads. Since no abrupt discontinuity of the shear can occur at the transverse walls, the computed "stepped" curve of the mean shear flows in the longitudinal direction must be smoothed out by a continuous curve in such a manner that the mean value within the bays remains the same. (See fig. 8a.) Similarly the stringer forces which are obtained by the statically indeterminate computation at the transverse stiffener walls must be joined by a corresponding continuous curve. It is possible, however, to consider the stiffness of the bulkheads as distributed over the shell length through the flexural stiffness of the longitudinal stiffeners and in the solutions for the redundancies pass to the limit of bulkheads spaced infinitely close. There is then obtained directly for the stringer forces a continuous axial distribution. For the case of loading of a very long shell composed of equal panels by means of characteristic force groups at an end bulkhead, these solutions are given in IV, 2, in the table of formulas 3. From the shear distribution the peripheral forces in the skin may be determined as has been done in the work of E. Schapitz and G. Krümling, cited in reference 1, starting from the experimentally determined longitudinal stresses.

## IV. SIMPLIFICATIONS FOR THE CASE OF EQUAL PANELS

### 1. Displacement Coefficients and Elasticity Equations

Of the simplifications of the statically indeterminate computation with uniform dimensions over the shell length, the case of geometrically and structurally equal panels will be considered. Systems of stepped dimensions may be approximated to the above case if the panel dimensions in the neighborhood of disturbance positions (cut-outs, deformation restraints) are used as a basis for the computation.

In the notation the subscript $k$ is dropped; the equations for the stress distributions have the same forms - the $w$ values according to (12), (14), (15), from which the displacement coefficients are formed being the same for all $k$'s. It is convenient for the computation to break up $w_L^{(\mu,\nu)}$ and $w_B^{(\mu,\nu)}$ as well as $w_S^{(\mu,\nu)}$ in (15) into a coefficient $\kappa^{(\mu,\nu)}$ and a factor $\omega$ which depends only on the system dimensions. In addition, we write:

$$w_L^{(\mu,\nu)} = \frac{4}{aE}\,\kappa_L^{(\mu,\nu)}\,\omega_L;\quad w_B^{(\mu,\nu)} = \frac{1}{aE}\,\kappa_B^{(\mu,\nu)}\,\omega_B$$

$$w_S^{(\mu,\nu)} = \frac{1}{4a\,E}\,\kappa_S^{(\mu,\nu)}\,\omega_S \tag{28}$$

with $$\omega_L = \frac{1}{8};\quad \omega_B = \frac{U}{\frac{G}{E}\,s^*};\quad \omega_S = \frac{UF^2}{aJ_S}$$

$$\left(s^* = \text{a mean skin thickness, e.g., } s^* = \frac{1}{U}\sum_{j=1}^{m} s_j b_j\right) \tag{29}$$

Then, according to (12) and (14):

$$\kappa_L^{(\mu,\nu)} = \frac{1}{4} \sum_{j=1}^{m} \frac{a^2}{F_j} P_j^{(\mu)} P_j^{(\nu)}$$

$$\kappa_B^{(\mu,\nu)} = \sum_{j=1}^{m} \frac{b_j}{U} \frac{s^*}{s_j} (a\ t_j^{(\mu)})\ (a\ t_j^{(\nu)}) \qquad (30)$$

For flexurally stiff bulkhead rings $\kappa_S^{(\mu,\nu)}$ is obtained from (16) as:

$$\kappa_S^{(\mu,\nu)} = \frac{4a}{UF^2}\left[ \oint B^{(\mu)} B^{(\nu)} du_S + i_S^2 \oint N^{(\mu)} N^{(\nu)} du_S \right] \qquad (30')$$

($t_j^{(\mu)}$ and $t_j^{(\nu)}$ are the shear flows in the $k^{th}$ panel due to $X_{\mu,k} = 1$ and $X_{\nu,k} = 1$, respectively; $B^{(\mu)}$, $N^{(\mu)}$ and $B^{(\nu)}$, $N^{(\nu)}$ are the bending moment and normal forces in the $k^{th}$ bulkhead due to these shears.)

The redundancies at the $k^{th}$ bulkhead are similarly combined with those lying to the right and to the left:

$$\delta_{k,k+1}^{(\mu,\nu)} = \delta_{k,k-1}^{(\mu,\nu)}; \quad \delta_{k,k+2}^{(\mu,\nu)} = \delta_{k,k-2}^{(\mu,\nu)}$$

According to (17a,b,c), the displacement coefficients using the above notation are:

$$\left.\begin{aligned} a\,E\,\delta_{k,k}^{(\mu,\nu)} &= 8\kappa_L^{(\mu,\nu)} \omega_L + 2\kappa_B^{(\mu,\nu)} \omega_B + \frac{3}{2} \kappa_S^{(\mu,\nu)} \omega_S \\ a\,E\,\delta_{k,k-1}^{(\mu,\nu)} &= 2\kappa_L^{(\mu,\nu)} \omega_L - \kappa_B^{(\mu,\nu)} \omega_B - \kappa_S^{(\mu,\nu)} \omega_S \\ a\,E\,\delta_{k,k-2}^{(\mu,\nu)} &= \qquad\qquad \frac{1}{4} \kappa_S^{(\mu,\nu)} \omega_S \end{aligned}\right\} \qquad (31)$$

Similarly simple expressions are obtained for the load coefficients. In applying the force groups $X_{\mu,o} = 1$ of the form of redundancies at the bulkhead 0, they may be expressed in terms of the displacement coefficients of the redundancies since

$$\delta_{1,0}^{(\mu,\nu)} = \delta_{k,k-1}^{(\mu,\nu)} + \delta_{k,k-2}^{(\mu,\nu)}; \quad \delta_{2,0}^{(\mu,\nu)} = \delta_{k,k-2}^{(\mu,\nu)} \tag{32}$$

if the end bulkheads have the same elasticity as the intermediate stiffeners. If the end bulkheads 0 and n, on the contrary, are very stiff compared to the intermediate bulkheads (rigid in the limiting case), then the load coefficients are:

$$\delta_{1,0}^{(\mu,\nu)} = \delta_{k,k-1}^{(\mu,\nu)} + 2\delta_{k,k-2}^{(\mu,\nu)}; \quad \delta_{2,0}^{(\mu,\nu)} = \delta_{k,k-2}^{(\mu,\nu)} \tag{33}$$

In this case the following displacement coefficients of the redundancies also change:

$$\delta_{1,1}^{(\mu,\nu)} = \delta_{n-1,n-1}^{(\mu,\nu)} = \delta_{k,k}^{(\mu,\nu)} - \delta_{k,k-2}^{(\mu,\nu)} \tag{34}$$

The load coefficients for the moment coefficients $M_k$ at the bulkheads are according to (20), since $t_{j,k+1}^{(\mu)} = - t_{j,k}^{(\mu)} = - t_j^{(\mu)}$:

$$\delta_{k,0}^{(\mu)} = \frac{M_k}{2F} \sum_{j=1}^{m} \frac{b_j \ (a \ t_j^{(\mu)}}{s_j \ G} \tag{35}$$

With bays of equal dimensions therefore, the load coefficients of the redundancies vanish at the unloaded transverse stiffener walls. The same holds true for the bending under transverse forces.

The elasticity equations (21) may under these simplifications be considered as a simultaneous system of $(m - 3)$ linear difference equations of the fourth order with constant coefficients whose solutions can be obtained without too much computation work in simple cases. (See reference 11.) In the case of orthogonal characteristic force groups (or force groups which affect one another to a negligible extent), this system breaks up into $(m - 3)$ independent difference equations of the fourth order of the form:

$$\delta_{k,k-2} X_{k-2} + \delta_{k,k-1} X_{k-1} + \delta_{k,k} X_k + $$
$$+ \delta_{k,k+1} X_{k+1} + \delta_{k,k+2} X_{k+2} = - \delta_{k,0} \tag{36}$$

where the shorter notation $X_k$, $\delta_{k,k}$ etc., has been written for $X_{\mu,k}$, $\delta_{k,k}^{(\mu,\mu)}$ etc. The solutions of these difference equations must, on account of the incompleteness of the first and last elasticity equations (see fig. 7) satisfy certain boundary conditions.

The obtaining of solutions in finite form is possible for a free bay system - i.e., for a shell with nonrestrained end bulkheads. In the case of an end bulkhead n restrained against deformation, the redundancies $X_n$ at the fixed end are determined, using the free bay system as a statically indeterminate principal system. If in the free system $X_k^{(o)}$ are the redundancies due to the external load, $X_k^{(n)}$ the redundancies due to $X_n = 1$, the final solutions are of the form:

$$X_k = X_k^{(o)} + X_n \; X_k^{(n)}$$

From the last elasticity equation, there then follows:

$$X_n = - \frac{\delta_{n,n-2} \; X_{n-2}^{(o)} + \delta_{n,n-1} \; X_{n-1}^{(o)} + \delta_{n,o}}{\delta_{n,n-2} \; X_{n-2}^{(n)} + \delta_{n,n-1} \; X_{n-1}^{(n)} + \delta_{n,n}} \tag{37}$$

The magnitude of the longitudinal forces at the fixed end restrained against deformation (fig. 1b) must be determined, for example, by a statically indeterminate computation. The redundancies $X_k^{(o)}$ are generally negligibly small in the case of uniform structure of the closed shell. The redundancies $X_k^{(n)}$ are computed according to the method given for the application of longitudinal forces. The redundancies $X_n$ at the restrained end can then be determined according to (37). From a knowledge of the force distribution on the application of concentrated forces, the main deviations from the elementary stress conditions can therefore be determined, and for this reason this loading condition will be considered in detail.

## 2. Solutions in Finite Form of the Difference Equations

To solve the elasticity equations (36) considered as symmetrical difference equations for uniform force groups:

$$X_{k-2} + 2\gamma X_{k-1} + 2\beta X_k + 2\gamma X_{k+1} + X_{k+2} = -\eta_k \tag{38}$$

$$(k = 1, 2, \ldots\ldots, n - 1)$$

with
$$2\gamma = \frac{\delta_{k,k-1}}{\delta_{k,k-2}};\quad 2\beta = \frac{\delta_{k,k}}{\delta_{k,k-2}};\quad \eta_k = \frac{\delta_{k,0}}{\delta_{k,k-2}} \tag{39}$$

we start with the general solution of the homogeneous equation ($\eta_k = 0$). An exponential substitution leads to a characteristic equation of the fourth degree. Using hyperbolic and circular functions, respectively, there is obtained the following result:

The general solution of the homogeneous equation is composed of four independent particular solutions whose form depends on the value $D = \frac{2(\beta - 1)}{\gamma^2}$. The case occurring most in practice, namely, $\gamma < 0$ is assumed; for $\gamma > 0$, the general solutions are to be multiplied by $(-1)^k$.

For $D > 1$, the solution of the homogeneous equation is:

$$X_k = C_1 \cosh k\psi \cos k\chi + C_2 \cosh k\psi \sin k\chi$$
$$+ C_3 \sinh k\psi \cos k\chi + C_4 \sinh k\psi \sin k\chi \tag{40}$$

The arguments $\psi$ and $\chi$ satisfy the boundary conditions:

$$\cosh\psi \cos\chi = \left|\frac{\gamma}{2}\right|;\quad \sinh\psi \sin\chi = \left|\frac{\gamma}{2}\right| \sqrt{D - 1} \tag{41}$$

whence follows:

$$\psi = \frac{1}{2}\cosh^{-1}(A + B) \qquad A = \frac{\beta - 1}{2} = \frac{D}{4\gamma^2}$$

with

$$\chi = \frac{1}{2}(\cos^{-1})(A - B) \qquad B = \sqrt{\left(\frac{\beta + 1}{2}\right)^2 - \gamma^2} \tag{42}$$

$(A+1)^2$

(Of the multiple-valued inverse trigonometric functions, only the positive principal values are taken.)

If $D < 1$, then in place of the circular functions, there occur the corresponding hyperbolic functions:

$$X_k = C_1 \cosh k \psi \cosh k \rho + C_2 \cosh k \psi \sinh k \rho + C_3 \sinh k \psi \cosh k \rho + C_4 \sinh k \psi \sinh k \rho \qquad (40')$$

with

$$\cosh \psi \cosh \rho = \left|\frac{\gamma}{2}\right|; \quad \sinh \psi \sinh \rho = \left|\frac{\gamma}{2}\right| \sqrt{1 - D} \qquad (41')$$

$$\psi = \frac{1}{2} \cosh^{-1} (A + B); \quad \rho = \frac{1}{2} \cosh^{-1} (A - B) \qquad (42')$$

(A and B as in (42))

For $D = 1$ (double root of the characteristic equation) $\chi$ and $\rho$, respectively, are equal to zero, and

$$\psi = \frac{1}{2} \cosh^{-1} (A + B) = \cosh^{-1} \left|\frac{\gamma}{2}\right|$$

In addition to $\cosh k \psi$ and $\sinh k \psi$, $k \cosh k \psi$ and $k \sinh k \psi$ are also solutions, so that the general solution of the homogeneous equation is:

$$X_k = C_1 \cosh k \psi + C_2 k \cosh k \psi + C_3 \sinh k \psi + C_4 k \sinh k \psi \qquad (40'')$$

The complete solution of the difference equation is then made up of the solution of the homogeneous equation with four arbitrary constants and an arbitrary particular solution of the nonhomogeneous equation. In the case of a system with unrestrained end, the following boundary conditions must, on account of the incompleteness of both the first and last elasticity equations, be satisfied:

$$X_0 = 0; \quad X_{-1} = 0; \quad X_n = 0; \quad X_{n+1} = 0 \qquad (43)$$

There are thus obtained four linear equations for the constants $C_1$, $C_2$, $C_3$, $C_4$ in the general solution, and after these have been computed the solution of the elasticity equations is obtained in finite form.

As an example, we consider the case of constant load coefficients $\eta_k = \eta$ = constant, as is the case in loading the cylindrical shell with constant moments at all transverse stiffeners and where there is no restraint on the deformation of the end sections. Further, let $\gamma$ be assumed as $< 0$ and $D > 1$. A particular solution of the nonhomogeneous difference equation is:

$$X_k^* = - \frac{\eta}{2 + 4\gamma + 2\beta}$$

to which must be added the solution (40). After determining the constants C from the boundary conditions (43) (this formal computation is omitted), there is obtained the solution:

$$X_k = X_k^* - \frac{X_k^*}{N^*} \Big[ \Big\{ \sinh (k+1)\psi \sin (n-k+1)\chi - \sinh k\psi \sin (n-k)\chi \Big\} + \Big\{ \sin (k+1)\chi \sinh (n-k+1)\psi - \sin k\chi \sinh (n-k)\psi \Big\} \Big]$$

where $N^* = \sinh (n+1)\psi \sin \chi + \sin (n+1)\chi \sinh \psi$

(For $D < 1$ the hyperbolic functions are to be substituted everywhere for the circular functions with the argument $\rho$ in place of $\chi$, while for $D = 1$, the functional symbol sin and the argument $\chi$ are to be dropped.)

The setting up of a solution of this type is possible for further simple loading conditions and it is also possible to take into account modified dimensions of the end bulkheads by corresponding end conditions. Since the general formulas are not very explicit, however, it is better in any individual case to use a numerical method, or in the case of a few equations, to solve directly by the well-known elimination or iteration method. For the limiting case of rigid bulkheads, solution formulas have been set up for a series of load conditions in the work cited under reference 6.

In applying independent force groups of the form of static redundancies at the bulkhead 0, the load coefficients different from zero of the first and second equations may be determined by the boundary conditions and simple solutions in finite form may be obtained. According to formulas (32), (33), (34), we must have

$$X_0 = 1; \quad X_{-1} = 1; \quad X_n = 0; \quad X_{n+1} = 0 \tag{44}$$

for the case of elastic end bulkheads, and

$$X_0 = 1; \; X_{-1} + X_{+1} = 2; \; X_n = 0; \; X_{n-1} + X_{n+1} = 0 \tag{45}$$

for the case of rigid end bulkheads.

By means of these conditions, the four arbitrary constants $C$ in the general solution of the homogeneous equation are determined. The final results are presented in the table of formulas 1. The solutions for all cases may be expressed in a general form through auxiliary functions $g_k$ and the conditions $X_0 = 1$; $X_n = 0$ may be immediately verified.

In the table of formulas 1, there are given in addition the solutions for the limiting case of rigid bulkhead walls. The difference equation then reads:

$$X_{k-1} + 2\alpha X_k + X_{k+1} = 0 \tag{46}$$

with $2\alpha = \dfrac{\delta_{k,k}}{\delta_{k,k-1}}$ and the end values $X_0 = 1$, $X_n = 0$.
For $|\alpha| \neq 1$, the equation has the general solution

$$X_k = (\pm 1)^k \left[C_1 \cosh k\varphi + C_2 \sinh k\varphi\right]$$

(upper sign if $\alpha < 0$) with the argument

$$\varphi = \cosh^{-1} |\alpha| \tag{47}$$

For $|\alpha| = 1$ the solution is:

$$X_k = (\pm 1)^k \left[C_1 + C_2 k\right]$$

There are thus obtained for the above boundary values the solutions:

$$X_k = (\pm 1)^k \frac{\sinh (n - k)\varphi}{\sinh n\varphi}$$

and $$(\pm 1)^k \left(1 - \frac{1}{n} k\right)$$

Form of solution: $X_k = (\pm 1)^k \dfrac{g_k g_0 - g_{n-k} g_n}{g_0^2 - g_n^2}$; upper sign if $\gamma < 0$ or $\alpha < 0$; lower sign if $\gamma > 0$ or $\alpha > 0$

| Case | Intermediate bulkheads elastic, end bulkheads: elastic | rigid |
|---|---|---|
| $D > 1$ $(A-B<1)$ | $g_k = \sin(k+1)\chi \sinh(n+1-k)\psi$ $\mp \sin k\chi \sinh(n-k)\psi$ | $g_k = \sin(k+1)\chi \sinh(n+1-k)\psi$ $\mp 2\sin k\chi \sinh(n-k)\psi + \sin(k-1)\chi \sinh(n-1-k)\psi$ |
| $D = 1$ $(A-B=1)$ | $g_k = (k+1)\sinh(n+1-k)\psi$ $\mp k \sinh(n-k)\psi$ | $g_k = (k+1)\sinh(n+1-k)\psi$ $\mp 2k \sinh(n-k)\psi + (k-1)\sinh(n-1-k)\psi$ |
| $D < 1$ $(A-B>1)$ | $g_k = \sinh(k+1)\rho \sinh(n+1-k)\psi$ $\mp \sinh k\rho \sinh(n-k)\psi$ | $g_k = \sinh(k+1)\rho \sinh(n+1-k)\psi$ $\mp 2\sinh k\rho \sinh(n-k)\psi + \sinh(k-1)\rho \sinh(n-1-k)\psi$ |

Limiting case of rigid bulkheads: $g_k = \sinh(n-k)\varphi$ for $|\alpha| \neq 1$; $g_k = n-k$ for $|\alpha| = 1$

**Table I.- Solutions of elasticity equations for applied characteristic force group $X_0 = 1$ for shells of finite lengths (end bulkhead freely deformable).**

| Case | Intermediate bulkheads, elastic, end bulkheads: elastic | rigid |
|---|---|---|
| $A-B<1$ | $X_k = (\pm 1)^k e^{-k\psi}\left[\cos k\chi + \frac{\cos\chi \mp e^{-\psi}}{\sin\chi}\sin k\chi\right]$ | $X_k = (\pm 1)^k e^{-k\psi}\left[\cos k\chi + \frac{\cosh\psi\cos\chi \mp 1}{\sinh\psi\sin\chi}\sin k\chi\right]$ |
| $A-B=1$ | $X_k = (\pm 1)^k e^{-k\psi}\left[1 + (1 \mp e^{-\psi})k\right]$ | $X_k = (\pm 1)^k e^{-k\psi}\left[1 + \frac{\cosh\psi \mp 1}{\sinh\psi}k\right]$ |
| $A-B>1$ | $X_k = (\pm 1)^k e^{-k\psi}\left[\cosh k\rho \frac{\cosh\rho \mp e^{-\psi}}{\sinh\rho}\sinh k\rho\right]$ | $X_k = (\pm 1)^k e^{-k\psi}\left[\cosh k\rho + \frac{\cosh\psi\cosh\rho \mp 1}{\sinh\psi\sinh\rho}\sinh k\rho\right]$ |

Limiting case of rigid bulkheads $(\omega_S = 0)$: $X_k = (\pm 1)^k e^{-k\varphi}$

Upper sign if $\varkappa_S\omega_S \geq 2\varkappa_L\omega_L - \varkappa_B\omega_B$; Lower sign if $\varkappa_S\omega_S < 2\varkappa_L\omega_L - \varkappa_B\omega_B$

Constants.

$A = \dfrac{8\varkappa_L\omega_L + 2\varkappa_B\omega_B + \varkappa_S\omega_S}{\varkappa_S\omega_S}$ $\qquad \psi = \frac{1}{2}\,\mathrm{arc\,cosh}(A+B)$

$B = \dfrac{4}{\varkappa_S\omega_S}\sqrt{3\varkappa_L\omega_L(\varkappa_L\omega_L + \varkappa_B\omega_B + \varkappa_S\omega_S)}$ $\qquad \chi = \frac{1}{2}\,\mathrm{arc\,cos}(A-B)$

$\rho = \frac{1}{2}\,\mathrm{arc\,cosh}(A-B)$

$|\alpha| = \left|\dfrac{\varkappa_B\omega_B + 4\varkappa_L\omega_L}{\varkappa_B\omega_B - 2\varkappa_L\omega_L}\right|$ $\qquad \varphi = \frac{1}{2}\,\mathrm{arc\,cosh}|\alpha|$

**Table II- Solutions of elasticity equations for applied characteristic force group $X_0 = 1$ for infinitely long shells.**

| Case | Intermediate bulkheads elastic, end bulkheads: elastic | rigid |
|---|---|---|
| $\bar A - \bar B < 0$ | $X(x) = e^{-\bar\psi x}\left[\cos\bar\chi x + \frac{\bar\psi}{\bar\chi}\sin\bar\chi x\right]$ | $X(x) = e^{-\bar\psi x}\left[\cos\bar\chi x + \frac{1}{2}\left(\frac{\bar\psi}{\bar\chi} - \frac{\bar\chi}{\bar\psi}\right)\sin\bar\chi x\right]$ |
| $\bar A - \bar B = 0$ | $X(x) = e^{-\bar\psi x}\left[1 + \bar\psi x\right]$ | $X(x) = e^{-\bar\psi x}\left[1 + \frac{1}{2}\bar\psi x\right]$ |
| $\bar A - \bar B > 0$ | $X(x) = e^{-\bar\psi x}\left[\cosh\bar\rho x + \frac{\bar\psi}{\bar\rho}\sinh\bar\rho x\right]$ | $X(x) = e^{-\bar\psi x}\left[\cosh\bar\rho x + \frac{1}{2}\left(\frac{\bar\psi}{\bar\rho} + \frac{\bar\rho}{\bar\psi}\right)\sinh\bar\rho x\right]$ |

Limiting case of rigid bulkheads: $X(x) = e^{-\bar\varphi x}$

Constants

$$\bar A = \frac{\varkappa_B\,\omega_B}{\varkappa_S\,\bar\omega_S} \qquad \bar\psi = \sqrt{\bar A + \bar B}$$

$$\bar B = \sqrt{\frac{4\,\bar\varkappa_L}{\varkappa_S\,\bar\omega_S}} \qquad \bar\chi = \sqrt{\bar B - \bar A}$$

$$\bar\varkappa_L = \frac{\varkappa_L}{a^2},\quad \bar\omega_S = \frac{U f^2}{I_S}\,(= a^2\omega_S) \qquad \bar\rho = \sqrt{\bar A - \bar B}$$

$$I_S = \frac{1}{a}J_S \qquad \bar\varphi = 2\sqrt{\frac{\bar\varkappa_L}{\varkappa_B\,\omega_B}}$$

**Table III.- Distribution of force group X(0) = 1 for infinitely long shell with bulkheads lying infinitely close.**

| $\omega$ - Values | $\omega_L = \frac{1}{3}$ ; $\omega_B = \frac{U}{G/E \cdot s^*}$ ; $\omega_S = \frac{U f^2}{a\,J_s}$ $\left(s^* = \frac{s_1 + s_2}{2}\right)$ | |
|---|---|---|
| $\varkappa$ - Values | Arbitrary doubly symmetrical cross section. | Circular cross section with uniform stringer spacing |
| Due to $X_K = 1$ | $\varkappa_L = \frac{a^2}{F_1}$<br>$\varkappa_B = \frac{4b_1}{U}\left(\frac{f_2}{f_1+f_2}\right)^2\frac{s^*}{s_1} + \frac{4b_2}{U}\left(\frac{f_1}{f_1+f_2}\right)^2\frac{s^*}{s_2}$<br>$\varkappa_S = \frac{4a^2}{Uf^2}\left[\oint B_X^2 du_s + i_s^2\oint N_X^2 du_s\right]$ | $\varkappa_L = \frac{a^2}{F_1}$ ; $\varkappa_B = \frac{s^*}{8}\left(\frac{1}{s_1} + \frac{1}{s_2}\right)$<br>$\varkappa_S = \lambda_S\left[2{,}0833 + 3{,}6818\,\lambda_L^2\left(\lambda_S^2 + \frac{i_s^2}{r^2}\right) - 5{,}5370\,\lambda_L\lambda_S\right]10^{-2}$ |
| Due to $Y_K = 1$ | $\varkappa_L = \frac{a^2}{2F_0} + \left(\frac{y_0}{y_1}\right)^2\frac{a^2}{4F_1}$<br>$\varkappa_B = \frac{b_1}{U}\frac{s^*}{s_1} + \frac{b_2}{U}\left(\frac{y_0}{y_1} - 1\right)^2\frac{s^*}{s_2}$<br>$\varkappa_S = \frac{4a^2}{Uf^2}\left[\oint B_Y^2 du_s + i_s^2\oint N_Y^2 du_s\right]$ | $\varkappa_L = \frac{a^2}{2}\left(\frac{1}{F_0} + \frac{1}{F_1}\right)$; $\varkappa_B = \frac{s^*}{8}\left(\frac{1}{s_1} + \frac{3 - 2\sqrt{2}}{s_2}\right)$<br>$\varkappa_S = \lambda_S\left[46{,}2136 + 57{,}4391\,\lambda_L^2\left(\lambda_S^2 + \frac{i_s^2}{r^2}\right) - 103{,}0099\,\lambda_L\lambda_S\right]10^{-4}$ |

**Table IV.- Displacement factors for six-stringer shells of equal bays.**

The solution formulas simplify considerably for the limiting case of infinite shell length - i.e., $n \rightarrow \infty$ for constant finite bulkhead spacing a. The results for this limiting condition are presented in the table of formulas 2. These solutions may be applied to shells of finite lengths greater than their perimeters, without appreciable error, as is shown with the aid of examples in section VI, 1 below. The solution becomes particularly simple for the limiting case of rigid bulkheads. If in $\alpha$, $\beta$, $\gamma$, the $\delta$ values are expressed in terms of the $\kappa$ and $\omega$ values corresponding to (31), these coefficients need not be further specially determined - the constant values of the solution formulas being computed directly from the $\kappa$ and $\omega$ values, as indicated in the table of formulas 2. The arguments $\psi, \chi$ are taken from tables of functions[7] with the natural numbers as argument. The condition

$$D \gtrless 1 \quad \text{and} \quad A - B \lessgtr 1$$

is equivalent to

$$12\ (\kappa_L\ \omega_L)\ (\kappa_S\ \omega_S) \gtrless (2\kappa_L\ \omega_L - \kappa_B\ \omega_B)^2$$

and the condition $\gamma \gtrless 0$, according to which the sign in the solution formulas is determined becomes, in terms of the $\kappa$ and $\omega$ values:

$$\kappa_S\ \omega_S \lessgtr 2\kappa_L\ \omega_L - \kappa_B\ \omega_B$$

For some simple shell shapes the $\kappa$ and $\omega$ values are given in table of formulas 4.

A further limiting case of importance is that of infinitely close bulkhead spacing for which the total transverse stiffness of the system is assumed to remain unchanged. If the bulkheads, for example, are rings with the bending stiffness $EJ_S$, the bending stiffness $EI_S = \frac{1}{a} EJ_S$ per unit length in the axial direction is to remain constant. The difference equation then goes over into a corresponding differential equation, the independent force group $X_k$ becoming a continuous function $X_{(x)}$ of the axial coordinate $x$ measured from the loading side. Again

[7]For example, the tables of circular and hyperbolic function of K. Hayashi, or "Hutte," vol. I.

we omit the computational details of passing to the limit and present the final solutions in the table of formulas 3. The case corresponding to $\gamma > 0$ drops out, the cases $\bar{A} - \bar{B} \lessgtr 0$ correspond to $D \gtrless 1$, the arguments are determined by square-root expressions, and in place of the $\kappa$ and $\omega$ values there enter corresponding $\bar{\kappa}$ and $\bar{\omega}$ values which no longer contain the bulkhead spacing a.

If the shear deformations are neglected $(\kappa_B\,\omega_B = 0)$, then $\bar{\Psi} = \bar{X}$, and the solutions are:

$$X_{(x)} = e^{-\bar{\Psi} x} (\cos \bar{\Psi} x + \sin \bar{\Psi} x)$$

for elastic end bulkhead, and

$$X_{(x)} = e^{-\bar{\Psi} x} \cos \bar{\Psi} x$$

for rigid end bulkhead, with

$$\bar{\Psi} = \sqrt[4]{\frac{4\bar{\kappa}_L}{\kappa_S\,\bar{\omega}_S}}$$

They have the same form as the function $f_e(x)$ for the simple disturbance loads in the work of Wagner and Simon, mentioned in reference 3. The argument values likewise agree for corresponding stress distribution, as may be shown by the example of a box beam of sides b and c and of constant wall thickness s. From the work cited under reference 7 (pp. 76-77), there is obtained for equal panels:

$$\kappa_L\,\omega_L \doteq \frac{2a^2}{s\,(b + c)}; \quad \kappa_S\,\omega_S = \frac{b^2\,c^2\,(b + c)}{24a\,J_S}$$

so that

$$\bar{\kappa}_L = \frac{6}{s\,(b + c)}; \quad \kappa_S\,\omega_S = \frac{b^2\,c^2\,(b + c)}{24\,I_S}$$

We then have:

$$\bar{\Psi} = 8\sqrt{3}\,\frac{b + c}{\sqrt{b\,c}}\,\sqrt[4]{\frac{I_S}{s\,U^6}} = 13.856\left(\sqrt{\frac{b}{c}} + \frac{1}{\sqrt{\frac{b}{c}}}\right)\sqrt[4]{\frac{I_S}{s\,U^6}}$$

in agreement with the corresponding $\omega_e$ value, according to table 2 of reference 3.

## V. CONSIDERATION OF SEVERAL IMPORTANT SYSTEMS

For several simple shell shapes with few stringers and whose cross sections are symmetrical about two perpendicular axes, there will be computed the stresses and displacement values. The transverse stiffeners are taken to be flexurally stiff rings of constant section over the entire circumference and with radii of gyration small compared to the diameters. Shells whose cross sections deviate little from the condition of double symmetry (monocoque fuselages) may be treated in approximately the same manner as these simple systems.

### 1. Four-Stringer Shells

The simplest statically indeterminate shape of shell is the 4-stringer system with intermediate transverse stiffeners. The stringers are assumed not to lie on the axis of symmetry. As redundancies, there are set up at each intermediate transverse stiffener $k$ and at the end restraint, a force group $X_k$ symmetrical with respect to the center and denoted briefly as a "convexing force group." The unit state $X_k = 1$ is shown in figure 9, and the reactions of the constant shear flows $t_1$ and $t_2$ on stringers and rings, which for the present are assumed to be attached to the skin, are indicated for the $k^{th}$ bay.

In the stringers according to (3) are set up axial forces which drop off linearly from 1 to 0, and for the shear flows there is obtained according to (4) and (5):

$$\left.\begin{aligned} t_{1,k} &= \frac{1}{a_k}\,\frac{F_2}{F_1+F_2}; \quad t_{2,k} = -\frac{1}{a_k}\,\frac{F_1}{F_1+F_2} \\ t_{1,1+1} &= -\frac{1}{a_{k+1}}\,\frac{F_2}{F_1+F_2}; \quad t_{2,k+1} = \frac{1}{a_{k+1}}\,\frac{F_1}{F_1+F_2} \end{aligned}\right\} \qquad (48)$$

The stresses of the flexurally stiff, threefold statically indeterminate bulkhead rings (On the computation of bulkhead rings, see among others, reference 12) may in the case here considered of double symmetry and loading symmetrical about the center, be obtained without statically indeterminate computations, since at the four diametral points on the axes the bending moments and the normal forces vanish and the transverse forces $Q_y$ and $Q_z$ at these points are obtained from the equilibrium conditions. Let us consider the $k^{th}$ ring under tangential loading by the shears of the $k^{th}$ bay (fig. 9b). Substituting from (48), we have:

$$\left.\begin{aligned} Q_{y,k} &= \frac{r_y}{a_k}\left(\frac{y_1}{r_y} - \frac{F_2}{F_1 + F_2}\right) \\ Q_{z,k} &= -\frac{r_z}{a_k}\left(\frac{z_1}{r_z} - \frac{F_1}{F_1 + F_2}\right) \end{aligned}\right\} \quad (49)$$

With the aid of the above equations, the distribution of the bending moments, and the normal and the transverse forces over the entire ring may readily be obtained. The bending moment $B_k$, for example, at a point $y_{S,k}$, $z_{S,k}$ of the neutral axis of the ring section is:

$$\left.\begin{aligned} B_k &= Q_{y,k}\, z_{S,k} - t_{1,k}\,(2f_1) \quad \text{in peripheral range } b_1 \\ B_k &= -Q_{z,k}\, y_{S,k} + t_{2,k}\,(2f_2) \quad \text{in peripheral range } b_2 \end{aligned}\right\} (50)$$

(Bending moment positive if the extreme outer ring fibers are under pressure.) $f_1$ and $f_2$ are the hatched areas indicated in figure 9b. The ring section may be different for each bulkhead, and hence also the coordinates $y_{S,k}$, $z_{S,k}$ of the neutral axis. The coordinates $y_1$, $z_2$ of the stringer center of gravity are assumed constant over the entire shell length. The $k^{th}$ bulkhead is in a similar manner loaded by the shears of the $(k+1)^{th}$ panel; it is only necessary to substitute $a_{k+1}$ for $a_k$.

If the shell cross section is an ellipse: $y = r_y \cos \varphi$, $z = r_z \sin \varphi$; $y_1 = \lambda_L r_y \cos \varphi_1$; $z_1 = \lambda_L r_z \sin \varphi_1$, the coordinates of the stringer centers of gravity and the shape of the neutral axis similar to that of the circumference around the skin: $y_{S,k} = \lambda_{S,k} r_y \cos \varphi$; $z_{S,k} =$

$\lambda_{S,k}\ r_z \sin\varphi$[8] (where $\lambda_L$, $\lambda_{S,k}$ are absolute numbers less than 1), there is obtained, after a short computation for the bending moments:

$$\left.\begin{aligned} B_k &= \frac{r_y r_z}{a_k}\left[(\lambda_L\ \lambda_{S,k} \cos\varphi_1) \sin\varphi - \left(1 - \frac{2\varphi_1}{\pi}\right)\varphi\right] \\ &\qquad \text{in peripheral range } b_1 \\ B_k &= \frac{r_y r_z}{a_k}\left[(\lambda_L\ \lambda_{S,k} \sin\varphi_1) \cos\varphi - \frac{2\varphi_1}{\pi}\left(\frac{\pi}{2} - \varphi\right)\right] \\ &\qquad \text{in peripheral range } b_2 \end{aligned}\right\} \quad (51)$$

For a circular cylinder, $\varphi$ and $\varphi_1$ are the angles subtended at the center (fig. 10), and

$$r_y = r_z = r; \quad \lambda_{S,k} = \frac{r_{S,k}}{r}; \quad \lambda_L = \frac{r_L}{r}$$

In this case the normal forces $N_k$ may readily be obtained:

$$\left.\begin{aligned} N_k &= \frac{r}{a_k}\,\lambda_L \cos\varphi_1 \sin\varphi \quad \text{for} \quad 0 \le \varphi \le \varphi_1 \\ N_k &= \frac{r}{a_k}\,\lambda_L \sin\varphi_1 \cos\varphi \quad \text{for} \quad \varphi_1 \le \varphi \le \frac{\pi}{2} \end{aligned}\right\} \quad (52)$$

If the rings are not attached to the skin the same "mean" shear stress distribution of the extended shell model is assumed according to (48), while the rings are loaded by distributed radial forces. According to (27), section III, 4, this loading $R_k$ of the $k^{th}$ ring per unit distance around the circumference and resulting from the circumferential forces in the $k^{th}$ and $(k + 1)^{th}$ bays due to $X_k = 1$, is:

$$R_k = \begin{cases} -\left(\dfrac{1}{a_k} + \dfrac{1}{a_{k+1}}\right)\left(1 - \dfrac{2\varphi_1}{\pi}\right)\varphi & \text{for} \quad 0 \le \varphi \le \varphi_1 \\ -\left(\dfrac{1}{a_k} + \dfrac{1}{a_{k+1}}\right)\dfrac{2\varphi_1}{\pi}\left(\dfrac{\pi}{2} - \varphi\right) & \text{for} \quad \varphi_1 \le \varphi \le \dfrac{\pi}{2} \end{cases}$$

[8]This assumption holds true approximately for constant ring section, since the neutral axis is displaced inward for strong curvature. (See reference 13.)

In the first and third quadrants, the radial load is therefore compressive while in the second and fourth quadrants, it is tensile. $E_k$ is split up into two parts proportional to $\frac{1}{a_k}$ and $\frac{1}{a_{k+1}}$. The bending moments and normal forces in the $k^{th}$ bulkhead due to the contribution from the $k^{th}$ bay are then:

$$\left.\begin{aligned} B_k &= \frac{r^2}{a_k}\left[(\lambda_L\,\lambda_{S,k}\cos\varphi_1)\sin\varphi - \lambda_{S,k}\left(1-\frac{2\varphi_1}{\pi}\right)\varphi\right] \\ &\qquad\qquad \text{for } 0 \leq \varphi \leq \varphi_1 \\ B_k &= \frac{r^2}{a_k}\left[(\lambda_L\,\lambda_{S,k}\sin\varphi_1)\cos\varphi - \lambda_{S,k}\frac{2\varphi_1}{\pi}\left(\frac{\pi}{2}-\varphi\right)\right] \\ &\qquad\qquad \text{for } \varphi_1 \leq \varphi \leq \frac{\pi}{2} \end{aligned}\right\} \quad (53)$$

$$\left.\begin{aligned} N_k &= \frac{r}{a_k}\left[(\lambda_L\cos\varphi_1)\sin\varphi - \left(1-\frac{2\varphi_1}{\pi}\right)\varphi\right] \\ &\qquad\qquad \text{for } 0 \leq \varphi \leq \varphi_1 \\ N_k &= \frac{r}{a_k}\left[(\lambda_L\sin\varphi_1)\cos\varphi - \frac{2\varphi_1}{\pi}\left(\frac{\pi}{2}-\varphi\right)\right] \\ &\qquad\qquad \text{for } \varphi_1 \leq \varphi \leq \frac{\pi}{2} \end{aligned}\right\} \quad (54)$$

The bending moments differ from the values in the case of tangential loading according to (51) only through the "excentricity" $\lambda_{S,k} = \frac{r_{S,k}}{r}$, occurring as a factor of the second member, and they are therefore larger since $\lambda_{S,k} < 1$. The normal forces, however, are considerably smaller, as may be seen by comparison with (52).

The displacement coefficients, corresponding to (17a, b, c), are made up of the w values, defined in (12), (14), (15):

$$w_{L,k} = \frac{a_k}{3}\,\frac{4}{E\,F_{1,k}} \quad (55)$$

$$w_{B,k} = \frac{4}{a_k G}\left[b_1\left(\frac{F_2}{F_1+F_2}\right)^2\frac{1}{s_{1,k}} + b_2\left(\frac{F_1}{F_1+F_2}\right)^2\frac{1}{s_{2,k}}\right] \quad (56)$$

The value of $\kappa_{S,k}$ according to (16) for the case of cross sections of general shape, may be determined only by numerical integration - for example, by the Simpson rule. For the circular bulkhead the integrals may be explicitly evaluated. In the case of tangential shear load with the stresses according to (51) and (52), there is obtained:

$$\kappa_{S,k} = \lambda_{S,k} \frac{8}{\pi^4} \left[\frac{2}{3} \varphi_1{}^2 \left(\frac{\pi}{2} - \varphi_1\right)^2 + \lambda_L{}^2 \left(\lambda^2_{S,k} + \frac{i^2_{S,k}}{r^2}\right) \frac{\pi}{2} \right.$$
$$\left\{\varphi_1 \cos^2 \varphi_1 + \left(\frac{\pi}{2} - \varphi_1\right) \sin^2 \varphi_1 - \frac{1}{2} \sin 2 \varphi_1\right\}$$
$$\left. - \lambda_L \lambda_{S,k} \left\{\pi \sin 2\varphi_1 - 4\varphi_1 \left(\frac{\pi}{2} - \varphi_1\right)\right\}\right] \tag{57}$$

With radial loading, however, when the rings are not attached to the skin, we have:

$$\kappa'_{S,k} = \lambda_{S,k} \frac{8}{\pi^4} \left(\lambda^2_{S,k} + \frac{i^2_{S,k}}{r^2}\right) \left[\frac{2}{3} \varphi_1{}^2 \left(\frac{\pi}{2} - \varphi_1\right)^2 \right.$$
$$+ \lambda_L{}^2 \frac{\pi}{2} \left\{\varphi_1 \cos^2 \varphi_1 + \left(\frac{\pi}{2} - \varphi_1\right) \sin^2 \varphi_1 - \frac{1}{2} \sin 2\varphi_1\right\}$$
$$\left. - \lambda_L \left\{\pi \sin 2\varphi_1 - 4\varphi_1 \left(\frac{\pi}{2} - \varphi_1\right)\right\}\right] \tag{58}$$

For the practical computation of these coefficients, there may be used the chart in figure 11, giving the auxiliary functions $K(\varphi,\lambda)$ and $\overline{K}(\varphi,\lambda)$[9] through which $\kappa_{S,k}$ and $\kappa'_{S,k}$ may be expressed in simple form. The dependence on the angle $\varphi_1$ may, with good approximation, be expressed by the following formula:

$$\kappa_{S,k}(\varphi_1) \approx \frac{1}{2}(1 - \cos 4\varphi_1)\, \kappa_{S,k}(45^\circ) \tag{59}$$

where $\kappa_{S,k}$, according to (57) and (58), respectively, on substituting the numerical values, is:

$$\kappa_{S,k}(45^\circ) = \lambda_{S,k} \left[2.0833 + 3.6818\, \lambda_L{}^2 \left(\lambda_{S,k} + \frac{i^2_{S,k}}{r^2}\right) - 5.5370\, \lambda_L \lambda_{S,k}\right] 10^{-2} \tag{60}$$

[9] $K(\varphi,\lambda)$, for example, is the value of $\kappa_{S,k}$ according to (57) for $\lambda_L = \lambda_{S,k} = \lambda$ and $\frac{i_{S,k}}{r} = 0$.

for the case of tangential loading, and

$$\kappa'_{S,k}\ (45^\circ) = \lambda_{S,k}\left(\lambda_{S,k}^2 + \frac{i_{S,k}^2}{r^2}\right)[2.0833 + 3.6818\ \lambda_L^2 - 5.5370\ \lambda_L]\ 10^{-2} \qquad (61)$$

for the case of radial loading.

These formulas for the $\kappa_{S,k}$ values of the circular ring may be used approximately for other ring shapes. For the "excentricities" $\lambda_{S,k}$ and $\lambda_L$ mean values are used; for $\varphi_1$ is substituted the angle which divides the quadrant in the same ratio as the stringer divides the circumferential quadrants of the actual shell $\left(\varphi_1 = \frac{\pi}{2}\frac{b_1}{b_1 + b_2}\right)$. A numerical example will make this approximate method clear. For an elliptical ring with ratio of axes $1 : \frac{1}{2}$ and portions of circumference $b_1 = b_2 = \frac{U}{8}$ the evaluation of $\kappa_{S,k}$ according to the Simpson rule, if the contribution of the normal forces is neglected, gives:

$$\kappa_{S,k} = \begin{cases} 0.000797 & \text{for } \lambda_L = \lambda_{S,k} = 0.95 \\ 0.002173 & \text{for } \lambda_L = \lambda_{S,k} = 1 \end{cases}$$

while the corresponding values for a circular bulkhead with $\varphi_1 = \frac{\pi}{4}$ are 0.000808 and 0.002282. The error for this narrow ellipse therefore amounts only to about 5 percent.

If the panels have the same dimensions axially, the subscript $k$ is dropped in the formulas and the displacement coefficients are then given in the form (31) with

$$\kappa_L = \frac{a^2}{F_1};\ \kappa_B = \frac{4b_1}{U}\left(\frac{F_2}{F_1 + F_2}\right)^2 \frac{s^*}{s_1} + \frac{4b_2}{U}\left(\frac{F_1}{F_1 + F_2}\right)^2 \frac{s^*}{s_2} \qquad (62)$$

($s^*$ = mean wall thickness)

and $\kappa_S$ according to (57) or (58).

For the case of loading by a "convexing" force group $X_0$ at the end bulkhead 0, the load coefficients are formed according to (18) or (32), (33) from the displacement coefficients of the redundancies. In the case of moments $M_k$ distributively applied in the elementary manner, the load coefficients from (6), (20), and (48) are:

$$\delta_{k,0} = - \frac{8}{GF^2} \left[ T_k \left( \frac{b_1 F_2}{s_{1,k}} - \frac{b_2 F_1}{s_{2,k}} \right) - T_{k+1} \left( \frac{b_1 F_2}{s_{1,k+1}} - \frac{b_2 F_1}{s_{2,k+1}} \right) \right] \tag{63}$$

## 2. Six-Stringer Shells

By assuming additional stringers at the diametral points of the vertical axis, we obtain a 6-stringer shell (figs. 3 and 12). In addition to force group 1, there are two other types of redundancies, of which only the simple symmetrical force group $Y_k$ (denoted in fig. 3 by $X_{3,k}$) is of importance. This force group occurs, for example, in bending about the z axis.

Figure 12 shows the unit state $Y_k = 1$. There is again a linearly decreasing force distribution in the stringers. On account of the symmetry about the y axis, there follows directly from the equilibrium at the stringer 0:

$$t_{1,k} = - \frac{1}{2a_k}; \quad t_{1,k+1} = + \frac{1}{2a_{k+1}} \tag{64}$$

and further, from (4):

$$t_{2,k} = + \frac{1}{2a_k} \left( \frac{y_0}{y_1} - 1 \right); \quad t_{2,k+1} = - \frac{1}{2a_{k+1}} \left( \frac{y_0}{y_1} - 1 \right) \tag{65}$$

The determination of the bulkhead stress requires in this case a simple, statically indeterminate computation. For a circular cylindrical shell with uniform stringer spacing $\left(\varphi_1 = \frac{\pi}{4}\right)$ and with equal distances of all stringer centers of gravity from the center, this computation has been carried out elsewhere. (See reference 5, p. 464.) The bending moments and the normal forces in the $k^{th}$ bulkhead produced by the shears of the $k^{th}$ bay, are:

$$B_k = \frac{r^2}{2a_k}\left[\varphi - \frac{\pi}{2}\left(1 - \frac{1}{2}\sqrt{2}\right) - \lambda_L \lambda_{S,k}\left(\sin\varphi - \frac{1}{2}\cos\varphi\right)\right] \quad \text{for} \quad 0 \leq \varphi \leq \frac{\pi}{4}$$

$$B_k = \frac{r^2}{2a_k}\left[(\sqrt{2} - 1)\left(\frac{\pi}{2} - \varphi\right) - \lambda_L \lambda_{S,k} \frac{1}{2}\cos\varphi\right] \quad \text{for} \quad \frac{\pi}{4} \leq \varphi \leq \frac{\pi}{2} \qquad (66)$$

$$N_k = -\frac{r}{2a_k}\lambda_L\left(\sin\varphi - \frac{1}{2}\cos\varphi\right) \quad \text{for} \quad 0 \leq \varphi \leq \frac{\pi}{4}$$

$$N_k = -\frac{r}{2a_k}\lambda_L \frac{1}{2}\cos\varphi \quad \text{for} \quad \frac{\pi}{4} \leq \varphi \leq \frac{\pi}{2} \qquad (67)$$

In a similar manner may be computed the stress condition of rings not attached to the skin if radial load according to (27) is assumed.

For the displacement coefficients in the form (17a,b, c), the $w$ values for stringers and sheet according to (64), (65) are immediately obtained:

$$w_{L,k} = \frac{a_k}{3}\left[\frac{2}{E\ F_{0,k}} + \left(\frac{y_0}{y_1}\right)^2 \frac{1}{E\ F_{1,k}}\right] \qquad (68)$$

$$w_{B,k} = \frac{1}{a_k}\left[\frac{b_1}{s_{1,k}} + \left(\frac{y_0}{y_1} - 1\right)^2 \frac{b_2}{s_{2,k}\ G}\right] \qquad (69)$$

The value of $\kappa_{S,k}$ in the case of general shape of cross section must again be determined by a numerical process. For a circular ring and with uniform stringer spacing $\left(\varphi_1 = \frac{\pi}{4}\right)$, the value of $\kappa_{S,k}$ may be computed on the basis of formulas (66), (67):

$$\kappa_{S,k} = \lambda_{S,k}\left[46.2136 + 57.4391\ \lambda_L^2\left(\lambda_{S,k}^2 + \frac{i_{S,k}^2}{r^2}\right) - 103.0099\ \lambda_L \lambda_{S,k}\right] 10^{-4} \qquad (70)$$

As in the case of the "convexing" force group, this value may also be used approximately for cross sections which deviate from the circular shape but where the stringer spacing is about uniform. In loading the bulkheads there are, in comparison with the convexing force group, twice as many zero positions of the bending moment. The contribution of the rings to the displacement coefficient thus becomes smaller as compared with the contributions from the stringers and shear sheets. The force distribution then depends very little on the stress or the stiffness of the transverse stiffener walls. As the following numerical examples show, for shell shapes of such dimensions as occur approximately in monocoque fuselages, the statically indeterminate computation for the simply symmetrical redundancies may be carried out approximately under the assumption of perfectly rigid transverse stiffeners. Thus there is avoided the inconvenient computation of the bulkhead coefficients and a corresponding simplification is gained in the solution of the elasticity equations.

With uniform dimensions of the bays, the displacement coefficients are formed out of the $\kappa$ and $\omega$ values. For the two redundancies $X_k = 1$ and $Y_k = 1$ these are, for convenience, collected in table of formulas 4. The displacement coefficients of the redundancies which occur in torsion and bending in the case of a 6-stringer shell with doubly symmetrical cross section, can then be obtained directly from (31). By substitution of the values in tables of formulas 1 and 2, there is obtained the distribution due to a force group $X_o$ and $Y_o$, respectively, applied at the end bulkhead.

The same redundant force groups hold also for an 8-stringer shell, obtained by adding two additional stringers at the diametral points of the horizontal axis. In addition, there are then obtained two doubly symmetrical force groups which for the loading case considered are without significance, and one force group simply symmetrical about the z-axis which is effective in bending about the y-axis.

### 3. Shells with More Stringers

For shells with more stringers with double symmetry of cross section, the redundancies must first be so determined that they have small effect on one another and their values may be determined with sufficient accuracy from the 5-member principal equations. In simple cases this may be done starting from the corresponding force groups with cyclical symmetry.

As an example, we consider a 12-stringer shell with equal panels. The force groups set up according to figure 4, break up into three "antisymmetrical," two doubly symmetrical, and four simply symmetrical groups, of which only the antisymmetrical and those simply symmetrical about the y-axis need be considered for the loading condition investigated (fig. 13a,b).

It is assumed that $b_1 = b_4$; $s_1 = s_4$; $b_2 = b_3$; $s_2 = s_3$; and $F_1 = F_3$ as, for example, is approximately the case with monocoque fuselages. If the effect of the bulkhead deformations - which with force groups of higher order is small compared to the effect of the deformations of the sheet and stringers - is neglected, then on account of the symmetry with respect to stringer 2 the antisymmetrical force groups $X_{1,k}$ and $X_{3,k}$ are orthogonal to $X_{2,k}$. The unknown forces c and d in $X_{1,k}$ and $X_{3,k}$ are now so determined that their mixed displacement coefficient contributions from stringers and sheet separately vanish. Then according to (31), any effect of different types of force groups occurs only through the bulkhead deformations, and therefore $w_L^{(1,3)} = w_B^{(1,3)} = 0$. According to (12) and (14), these conditions are:

$$\left.\begin{aligned} &\frac{2cd}{F_1} + \frac{1}{F_2} = 0 \\ &\frac{b_2}{s_2} - \frac{b_1}{s_1}(2c + 1)(2d - 1) = 0 \end{aligned}\right\} \qquad (71)$$

From those is obtained a quadratic equation for c or d, the two roots giving, except for a factor, the same force groups $X_{1,k}$ and $X_{3,k}$ but in different order. For the special further case of $b_1 = b_2 = b_3 = b_4 = \frac{U}{16}$, there is obtained for c and d:

$$\left.\begin{aligned} c &= \frac{1}{2F_2}\left[\sqrt{F_2^2 + F_1^2} - (F_2 - F_1)\right] \\ d &= \frac{1}{2F_2}\left[\sqrt{F_2^2 + F_1^2} + (F_2 - F_1)\right] \end{aligned}\right\} \qquad (72)$$

For the two force groups $Y_{1,k}$, $Y_{2,k}$ symmetrical about the vertical axis (fig. 13b), there are determined in similar manner the unknown forces c and d from the conditions that the mixed displacement contributions from stringers and sheet vanish separately. The effect through the bulkhead deformations is then in both cases so small, as shown by numerical examples, that their values may be computed from the uncombined principal equations.

The displacement coefficients from stringers and skin may for these force groups be simply given by the general formulas (12), (14), (17a,b,c). Whereas with groups of higher order the assumption of rigid bulkhead may in general be considered approximately true, in the case of the antisymmetrical force group $X_{1,k}$ the bulkhead coefficient must be taken into account. For ring bulkheads this coefficient can be determined approximately from the $\kappa_S$ values for the 4-stringer circular cylinder. The force group $X_{1,k}$ is composed of three simple convexing force groups at stringers 1, 2, 3. For these the $\kappa_S$ values are computed from the formulas for the circular ring bulkhead. On account of $b_1 = b_4$ the values for the groups at stringers 1 and 3 are equal to each other, so that only the two magnitudes

$$\kappa_S\ (\varphi_1) \quad \text{and} \quad \kappa_S\ (\varphi_2) \quad \text{for} \quad \varphi_1 = \frac{4b_1}{U},\ \varphi_2 = \frac{4\ (b_1+b_2)}{U}$$

need be taken from (57), (58), or figure 11. We then have approximately for the $\kappa_S$ value of the force group $X_{1,k}$:

$$\kappa_S \approx \kappa_S\ (\varphi_2) + 3c^2\ \kappa_S\ (\varphi_1) + 4c\sqrt{\kappa_S\ (\varphi_1)\ \kappa_S(\varphi_2)} \tag{73}$$

The accuracy will be checked with the aid of an example. For $\lambda_S = 0.938$, $\lambda_L = 0.9625$, $\frac{i_S}{r} \approx \frac{1}{25}$, $\varphi_1 = \frac{\pi}{8}$, $\varphi_2 = \frac{\pi}{4}$, there is obtained from figure 11: $\kappa_S\ (\varphi_2) = 8.52 \times 10^{-4}$, $\kappa_S\ (\varphi_1) = 4.29 \times 10^{-4}$. The approximate formula gives for $c = 0.236$: $\kappa_S \approx 14.94 \times 10^{-4}$, while the numerical integration according to the general formula (16) gives the more accurate value $15.17 \times 10^{-4}$. For practical computation the error is insignificant.

For shells with many stringers and with few axes of symmetry the setting up of force groups which are approximately orthogonal is very tedious. A fundamental method is given in the Appendix. By combining several longitudinal stiffeners into one fictitious stringer, it is generally possible to obtain the simple shell forms considered and thus a general view of the force distribution "in the large." The approximate computation may likewise be restricted to small ranges of the system in the neighborhood of "disturbance positions" (discontinuities in loading or dimensions, cutaways, etc.), in which case arbitrary linearly independent force groups are chosen as the static redundancies and the complete elasticity equations for these solved as in section III, 3.

## VI. NUMERICAL EXAMPLES AND COMPARISON WITH TEST RESULTS

### 1. Force Distribution in a Six-Stringer Shell

### Effect of the Various Stiffnesses

The effect of the various stiffnesses on the force distribution will be investigated with the aid of a simple example of a 6-stringer circular cylindrical shell of equal bays with uniform stringer distribution, and the reliability of approximate computations checked. The bulkhead rings are assumed to be attached to the skin.

Dimensions

$$r = 40 \text{ cm}; \quad r_S = 37 \text{ cm}; \; r_L = 38.5 \text{ cm}; \; a = 36 \text{ cm}$$

$$F_0 = 3.0 \text{ cm}^2; \; F_1 = 4.5 \text{ cm}^2$$

$$s_1 = s_2 = 0.06 \text{ cm}; \; J_S = 3.0 \text{ cm}^4; \; \frac{i_S}{r} \approx \frac{1}{25}$$

From table of formulas 4, the $\omega$ values, which are independent of the redundancies, are:

$$\omega_L = 0.333; \; \omega_B = 1.088 \times 10^4; \; \omega_S = 58.8 \times 10^6$$

Further:

$$\lambda_S = \frac{r_S}{r} = 0.925; \; \lambda_L = \frac{r_L}{r} = 0.9625$$

<u>a) Application of a convexing force group</u> $X_0 = 1$ (fig. 14).- According to table of formulas 4 and figure 11:

$$\kappa_L = 288; \; \kappa_B = 0.250; \; \kappa_S = 0.717 \times 10^{-3}$$

so that, $\kappa_L\,\omega_L = 0.96 \times 10^8$; $\kappa_B\,\omega_B = 27.2 \times 10^8$

$$\kappa_S\,\omega = 421.6 \times 10^8$$

From these there are obtained, according to the formulas in table 2, the values: $A = 1.147$; $B = 0.341$. We therefore have the case $(A - B) < 1$, with the upper sign. We then have:

$$\psi = \frac{1}{2}\cosh^{-1} 1.488 = 0.476,\ \chi = \frac{1}{2}(\cos^{-1})\ 0.806 = 0.317$$

The falling off to zero of the convexing force group $X_0 = 1$ for an infinitely long shell is then obtained for elastic and rigid end bulkheads, respectively, from the equations:

$$X_k = e^{-0.476k}\,(\cos 0.317\ k + 1.055\ \sin 0.317\ k)$$

and

$$X_k = e^{-0.476k}\,(\cos 0.317\ k + 0.390\ \sin 0.317\ k)$$

For the limiting case of perfectly rigid end bulkheads:

$$X_k = e^{-k\varphi} \text{ with } \varphi = \cosh^{-1}\frac{27.2+3.84}{27.2-1.92} = \cosh^{-1} 1.227 = 0.662$$

Figure 14 shows the numerical values of $X_k$ plotted against the shell length, the points being joined by straight lines. According to the shell model used the force distribution in one of the four loaded stringers is shown. For the case of elastic end bulkheads the values of $X_k$ are given for a 6-bay shell. Larger deviations from the values for the infinitely long shell occur only for small end forces, so that shells whose lengths are approximately equal to the circumferences, may be computed from the simpler formulas for the infinitely long shells.

The effect of the convexing force group will be smaller the stiffer the transverse stiffeners compared to the stringers and sheet. If the stringers, however, are very stiff $(\kappa_L\,\omega_L \approx 0)$, then the bulkhead effect vanishes; i.e., with finite length shells the forces in this case decrease linearly over the entire length.

A further limiting case is that of a sheet rigid in shear ($\kappa_B \omega_B \approx 0$) and the force distribution is shown in figure 14 (thin lines). The effect of the shear deformations on the force distribution appears to be greater the stiffer the bulkheads; a great stiffness in shear gives a more rapid rate of decrease in the forces. The continuous distribution of the forces computed according to table of formulas 3 gives at the transverse stiffeners about the same values as obtained according to table of formulas 2, and therefore has not been specially indicated.

For the statically indeterminate computation, it is important to know the effect of slight variations in the stiffnesses since the computation is made with estimated cross sections and skin thicknesses. Figure 15 shows the force distribution in a long shell with rigid end ring where the dimensions of bulkheads, stringers, and skin have been varied individually from the main assumption a). It may be seen that the force distribution is not very sensitive to those changes. From cases c) and d) it may be concluded that the total distribution after buckling of the skin does not change much, since the effects due to the smaller contribution of the skin in supporting the longitudinal forces and the decrease in the shear stiffness approximately offset each other. Upon this force distribution there is still to be superposed, however, the states of stress due to the tension fields themselves.

b) Application of a symmetrical force group $Y_0 = 1$ (fig. 16). The type of loading shown in figure 1a by a bending force group may, according to section II, 3 be reduced to a simply symmetrical force group $Y_0$. From table of formulas 4, we find:

$$\kappa_L = 360, \ \kappa_B = 0.1465, \ \kappa_S = 1.085 \times 10^{-5}$$

from which $A = 7.50$, $B = 5.77$. We thus have the case $A - B > 1$ with upper sign, and the falling off to zero for an infinitely long shell is aperiodic, for example, in the case of elastic end bulkheads, according to the formula:

$$Y_k = e^{-1.638k} (\cosh 0.573\ k + 1.612 \sinh 0.573\ k)$$

The results of computations for the different limiting cases of bulkhead stiffness are plotted in figure 16 against the shell length and the points joined by straight lines, the force distribution in stringer 0 being represented. There is also shown the limiting case ($\lambda_L = \lambda_S = 0$) for which $\kappa_S = 7.35 \times 10^{-5}$. This case may also be interpreted with respect to the original dimensions as a reduction in the bulkhead stiffness by about 1/7. There has further been computed the force distribution in the limiting case of sheet rigid against shear.

Since the force group $Y_0$ is of "higher order" (see II, 2) than the convexing group $X_0$, it reduces to zero over a shorter shell length. The force distribution depends only on the stiffness of the transverse stiffeners, particularly if there is a strong end bulkhead at the loading end. The dependence on the shear stiffness is considerably greater, however, than in the case of the convexing force group. As an approximating assumption for simplifying the computation, it is accordingly permissible to neglect the bulkhead deformations ($\kappa_S \omega_S = 0$) in the statically indeterminate computation for the simply symmetrical force groups.

## 2. Computation of a Test Shell

The computation procedure developed will be applied to a circular cylindrical 18-stringer shell, whose states of stress under an applied bending and convexing force group at 4 stringers, has been determined experimentally, as presented in detail in a simultaneously appearing paper by E. Schapitz and G. Krümling. (See reference 1.) The description of the shells and the method of conducting the tests may therefore here be disposed with. We further limit ourselves to the computation of the longitudinal stresses in the stringers before buckling of the skin takes place, and compare them with the measured values.

<u>a) Application of a bending force group.</u>- The actual arrangement of the stringers is shown in figure 17a for one quadrant. At stringers [7], [12], [3], [16][10], a pure bending moment about the transverse axis is applied by four concentrated forces P. We consider first the closed shell without the cutaway portion at the loading side and idealize the system to a 12-stringer shell by combining stringers [10] and [11] into one stringer 1 of double the cross-sectional area. For the skin thickness there is assumed a uniform value of $s = 0.06$ cm and half the skin

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[10]The stringer and bulkhead notations taken from the paper by E. Schapitz and G. Krümling are denoted by brackets.

strip of the neighboring sheet panels is added as effective width to each stringer section. These effective stringer sections are shown in figure 17a. We then have:

$$4\ (F_1\ y_1^2 + F_2\ y_2^2 + F_3\ y_3^2) = 2.50 \times 10^4\ \text{cm}^4$$

equal with sufficient approximation to the moment of inertia with respect to the transverse axis of the full shell in the last bays. The applied bending force group is split up as shown in figure 5. The component forces of the force group corresponding to the linear stress distribution over the cross section and having equal moment are:

$$P_1^{(o)} = 0.383\ P;\ P_2^{(o)} = 0.389\ P;\ P_3^{(o)} = 0.097\ P$$

As redundancies there occur only the simply symmetrical force groups $Y_{1,k}$ and $Y_{2,k}$ indicated in figure 13b. From the conditions for the vanishing of the mixed displacement coefficients from stringers and sheet there are obtained the values for the component forces c, d, $\bar{c}$, $\bar{d}$ shown in figure 17b (solution of a quadratic equation). The characteristic force group shown in figure 5 as the difference between the actual and linearly distributed moment is combined from the redundant longitudinal force groups in the following manner:

$$\text{Characteristic force group} = 0.431\ P\ (Y_{1,o} = 1)$$

$$- 0.262\ P\ (Y_{2,o} = 1)$$

The manner of decrease of $Y_{1,o} = 1$ and $Y_{2,o} = 1$ is computed under the approximating assumption of rigid bulkheads - which assumption is permissible according to the numerical example previously given. The bulkhead spacing is $a = 36$ cm and with the values determined according to (28) and (30):

$$\kappa_L\ \omega_L = 2.53 \times 10^2;\ \kappa_B\ \omega_B = 32.9 \times 10^2 \quad \text{for} \quad Y_{1,k} = 1$$

$$\kappa_L\ \omega_L = 2.86 \times 10^2;\ \kappa_B\ \omega_B = 12.84 \times 10^2 \quad \text{for} \quad Y_{2,k} = 1$$

there is obtained from table of formulas 2:

$$Y_{1,k} = e^{-1.001\ k};\ Y_{2,k} = e^{-1.898\ k}$$

From these values are obtained the stringer forces due to the characteristic force group, and by superposition with the above values of $P_1^{(o)}$, $P_2^{(o)}$, $P_3^{(o)}$, there is obtained the final force distribution plotted in figure 18.

In order to take into account the effect of the cutaway portion, this incomplete bay is separated from the full shell and a further simply symmetrical force group $\overline{Y}$ is assumed to act as static redundancy at stringers 2 and 3. For the condition $\overline{Y} = 0$ there is obtained in the full shell the force distribution previously computed, in the main spars of the cutaway portion the constant force $\frac{1}{2}$P. The stress condition $\overline{Y} = 1$ in the full shell is restricted approximately to the first bay. For the determination of $\overline{Y}$, the various skin thicknesses must be taken into account. Furthermore, the skin in the first bays does not contribute to the support of the longitudinal forces in the same degree as was previously assumed in the case of the complete shell. After computing the displacement coefficients according to the general formulas in section III, 2, using the values indicated in figure 18 for the dimensions (bulkheads assumed rigid), there is obtained $\overline{Y} = 0.0606$ P, and from this the force distribution given in figure 18.

For comparison with the measured stresses the mean stringer stresses must be determined from this force distribution. Since from bulkhead [d] on the stress distribution obtained is approximately linear (see fig. 18), it is possible from this position on to use the values for the stringer cross sections given in figure 17. At the cutaway portion and at the first bays, the skin contributed only imperfectly to the support. (See the paper by Schapitz and Krümling, reference 1.) At the cutaway portion the mean cross-sectional areas given in figure 18 apply; at bulkhead [f] in the full shell, we have approximately:

$$F_1 = 1.95 \text{ cm}^2; \; F_2 = 3.10 \text{ cm}^2; \; F_3 = 1.30 \text{ cm}^2$$

From these values up to those of figure 17a at bulkhead [d] a linear rate of increase is assumed. In figure 19 are plotted the stresses computed with these cross sections from the force distribution in figure 18, as well as the values given in figure 7 of the work by Schapitz and Krümling, the values being divided by an initial stress of 474 kg per cm² at the loading side. The stress distribution is in satisfactory agreement with the statically indeterminate computation. The assumption of rigid bulkheads in the computation leads to a more rapid state of decrease of the longitudinal stresses in the main spars and to a more rapid rate of rise in the intermediate stringers. Through the combination of stringers [10] and [11] the deformations of these stringers occurring in the first bays of the complete shell are not taken into account, thus resulting in a greater stress in stringer [11] than in stringer [10]. Further details of the stress condition (shears, bulkhead loading, etc.) are given in the experimental report referred to.

b) Application of a convexing force group.- For computing the force distribution resulting from the application of a "convexing" force group at the four main spars, the test specimen is idealized into a 12-stringer shell in a different manner. Since the intermediate stringers in the neighborhood of the upper and lower diametral points are hardly loaded this computation is restricted to the main spars and the adjacent intermediate stiffeners, and is based on a somewhat different circumferential spacing, as shown in figure 20a. The effect of the cutaway portion is neglected. For an approximate computation, we consider first a system of equal bays with skin thickness $s = 0.06$ cm and a bulkhead moment of inertia $J_S = 3\ cm^4$ (this is approximately the computed value for the section of the bulkhead rings [e] and [f] attached to the skin at the loading side.) To the main spars and intermediate stringer sections (see fig. 17a) there is added an effective skin strip of 12 and 14 cm, respectively, and there are thus obtained the effective stringer sections shown in figure 20.

Of the three antisymmetrical characteristic force groups at a 12-stringer shell (fig. 13a), only $X_{1,k}$ and $X_{3,k}$, due to the convexing force group, will be effective. For the component forces c and d, there are determined the values indicated in figure 20b from the formula (72) in section V, 3. The mixed displacement coefficients due to stringers and sheet then vanish. For the applied convexing force group $X_P$, the following relation is true:

$$X_P = 0.780\ P\ (X_{1,0} = 1) + 0.220\ P\ (X_{3,0} = 1)$$

The force distribution due to $X_{1,0} = 1$ and $X_{3,0} = 1$ is determined from independent difference equations, as shown in the first numerical example. The bulkhead coefficient $\kappa_S$ for the redundancy $X_{1,k}$ is already contained in the numerical example for the approximate formula (73). Figure 21 shows the results for different assumptions. The characteristic force group $X_{3,0} = 1$ decreases more rapidly than $X_{1,0} = 1$, since the former is of higher order. It may be determined with sufficient accuracy according to table of formulas 2 under the assumption of infinitely long shell and rigid transverse stiffeners. For the characteristic force group $X_{1,k}$ these assumptions are no longer admissible; table of formulas 1 must therefore be used and the effect of the bulkhead deformations taken into account. There was, furthermore, investigated the effect of the two redundancies due to the bulkhead deformations, by computing the mixed displacement coefficients and solving the complete elasticity equations. This effect is very small, the redundancies being practically orthogonal.

Since the variation of the redundancies $X_{1,k}$ depends essentially on the bulkhead stiffness, a more accurate computation was made for the actual dimensions "stepped" in the longitudinal direction. The effective moment of inertia of the bulkheads at the shell attachment was obtained by deflection measurements for the case of diametrically situated concentrated forces:

$$J_S = 6.5 \text{ cm}^4 \quad \text{for bulkhead} \quad [e] \quad \text{and} \quad [f]$$

$$J_S = 0.9 \text{ cm}^4 \quad \text{for bulkhead} \quad [a] \quad \text{to} \quad [d]$$

For the bulkheads [e] and [f] attached to the skin, the bulkhead coefficient under the assumption of tangential load is:

$$\kappa_S = 14.94 \times 10^{-4}$$

while for bulkheads [a] to [d] under the assumption of radial load (according to fig. 11 and approximate formula (73)):

$$\kappa_S{}' = 23.9 \times 10^{-4}$$

Furthermore, there was taken into account the variable skin thicknesses (0.08 cm in bay 1, 0.06 cm in bays 2 to 4). Under these assumptions there are obtained, according to the general procedure in section III, the values for $X_{1,k}$ plotted as a heavy line in figure 21. In spite of the weaker bulkheads at the end of the shell there is a more rapid rate of decrease than under the assumption of equal bays. This is due to the essentially smaller stiffness of the system from bays 2 to 4 as compared with the first bay.

With these more accurate values for $X_{1,k}$ and the approximate values of $X_{3,k}$ for rigid bulkheads, the force distribution shown in figure 22 was computed. For comparison with the experimental results there were further determined the stresses due to this force distribution. For this purpose there was assumed a linear rate of increase between the estimated reduced areas $F_1 = 1.35$ cm², $F_2 = 3.10$ cm² at bulkhead [f] and the stringer cross sections from bulkhead [d] on, according to figure 20. Since the effect of the cutaway portion is neglected, the comparison can only be made with the measured stress values in the main spars and in the intermediate stringers [11] and [17] (fig. 22). The experimentally determined stresses in the main spars decrease somewhat more rapidly than according to the computation, especially from the second bay on. As a result, the computed shears (see fig. 20 of the paper by Schapitz and Krümling) in the first bay are smaller, and in the last bays are greater than the experimentally determined values. The reason for this lies in the neglected bending stiffness of the main spars, as a result of which the bulkheads at the end of the shell receive additional stresses in the same direction as those from the shears. These bulkheads therefore act still more weakly than was assumed in the computation, and the reduction is concentrated more strongly on the stiffer first bay. In spite of the neglect of these factors, the method developed yields sufficiently accurate values.

Further conclusions as to the state of stress need not here be considered as these will be found in the work of Schapitz and Krümling, already referred to. The effect of modified test conditions may be roughly estimated from the investigations in the first numerical example. In particular, the tests confirm the conclusion drawn from figure 15, that the force distribution as a whole undergoes only a slight change after buckling of the skin.

## VII. SUMMARY

Longitudinally and transversely stiffened shells are, under the assumption of constant shear flow (shear times thickness) in each of the sheet panels, treated as systems with finite static redundancies. The procedure of the statically indeterminate computation is given for cylindrically shaped shells under conditions of loading by concentrated forces, moments, and transverse forces. As static redundancies longitudinal force groups are introduced at the intermediate transverse stiffeners and restraints between the bays, these force groups being so chosen that only groups of the same kind affect each other longitudinally - as a result of which the elasticity equations break up into independent 5-member partial systems. The manner of setting up of these "orthogonal" characteristic force groups is given for several simple shell shapes. Such force groups are in general possible only for a sufficiently large number of axes of symmetry, but otherwise only if the mutual effect through the deformations of the transverse walls or the shear sheets or also of the stringers is neglected. (See VIII, Appendix.) It is shown, with the aid of examples, that neglecting of the transverse stiffener deformations is admissible for the setting up of orthogonal force groups. The further result is obtained that the transverse stiffener deformations are of importance only for the distribution of the force groups which are in equilibrium over the entire shell circumference (for example, antisymmetrical force groups in torsion). For force groups of "higher" order - such, for example, as occur in bending - the transverse walls may be assumed as rigid.

In the case of equal bays, the elasticity equations are solvable in finite form. For the loading of the shell by concentrated forces, these solutions are given in tables of formulas. Similarly, there have been collected in one table, the displacement coefficients for 6-stringer shells of equal bays. With the aid of these tables the most important disturbances from the elementary force distribution, such as occur on application of concentrated forces at cutaway portions, may be approximately determined for the usual shell shapes.

The practical computation procedure is clarified with the aid of a simple numerical example, and the effect of the stiffnesses on the force distribution, investigated.

Further, there is computed according to the procedure proposed, the stringer stresses in a test shell with axial forces applied at four points and the stresses compared with the experimentally determined values. Satisfactory agreement is obtained between the computed and measured values. The same procedure, using the corresponding displacement coefficients, may be applied to other multi-stringer systems (multispar frame wings, airship hulls).

## VIII. APPENDIX

### Orthogonal Force Groups for Cylindrical Shells with Arbitrary Cross Sections

The setting up of uniform orthogonal characteristic force groups for shells of equal bays with arbitrary non-symmetrical cross section and many stringers, will be more closely investigated. If, in the mixed displacement coefficients of the force groups at the same transverse stiffeners, the contributions from the stringers, sheet, and bulkheads vanish separately, then according to (31) all mixed displacement coefficients of force groups of different kinds are equal to zero and conversely. It is therefore sufficient, for the setting up of orthogonal groups, to consider a system of two bays and at the intermediate transverse stiffener, to determine the redundancies $X_1$, $X_2$, ...., $X_{m-3}$ in such a manner that all mixed displacement contributions $\delta^{(L)}_{\mu,\nu}$, $\delta^{(B)}_{\mu,\nu}$, and $\delta^{(S)}_{\mu,\nu}$ vanish. There are $3\binom{m-3}{2} = \frac{3}{2}\,(m-3)\,(m-4)$ different portions of this kind.

Starting from arbitrary linearly independent force groups $\overline{X}_1$, $\overline{X}_2$ ...., $\overline{X}_{m-3}$ and forming from these by a linear transformation with determinant different from zero, the required force groups $X_1$, ...., $X_{m-3}$:

$$\overline{X}_\mu = \sum_{\nu=1}^{m-3} c_{\mu,\nu}\, X_\nu,\ (\mu = 1, 2, \ldots, m-3) \tag{74}$$

there are only $(m-3)^2 - (m-3) = (m-3)\,(m-4)$ essential constants $c_{\mu,\nu}$ available since each multiple of a force group may be chosen as a unit state. The $\frac{3}{2}\,(m-3)\,(m-4)$ equations of condition for the vanishing of

$\delta^{(L)}_{\mu,\nu}$, $\delta^{(B)}_{\mu,\nu}$, and $\delta^{(S)}_{\mu,\nu}$ cannot all, therefore, as a rule, be satisfied. In special cases - as, for example, with cyclic symmetry - these equations may be independent of one another, so that the free values of the above transformation are sufficient. In general, however, the equations are independent, and the orthogonality condition of the force groups cannot be obtained for arbitrary nonsymmetrical cross sections.

If the effect of one of the three stiffnesses of stringers, sheet, or bulkheads on the coupling of the different redundancies is neglected, then the $(n - 3)(n - 4)$ essential values $c_{\mu,\nu}$ of the transformation are sufficient for satisfying the remaining $2\binom{n-3}{2} = (n - 3)(n - 4)$ conditions. In the work of Wagner and Simon (reference 3), the effect of the shear stiffness is not taken into account, so that the determination of orthogonal independent stress groups is possible as characteristic solutions of a linear integral equation with symmetrical nucleus. As is shown by the numerical examples for stiffened shells, the effect of the bulkhead deformations on the force groups of higher order is small, so that it is convenient to neglect the effect of the bulkheads - i.e., to determine the force groups $X_\mu$ in such a manner that the mixed coefficients $\delta^{(L)}_{\mu,\nu}$ and $\delta^{(B)}_{\mu,\nu}$ vanish. The strain energy of the system composed of two panels may be expressed in terms of the redundancies $\overline{X}_1, \ldots, \overline{X}_{n-3}$ with the corresponding displacement coefficients $\overline{\delta}_{\mu,\nu}$. The quadratic portion in $\overline{X}$ consists, on neglecting the bulkhead deformations, of the two positive definite quadratic forms:

$$H^{(L)} = \frac{1}{2} \sum_{\mu,\nu} \overline{\delta}^{(L)}_{\mu,\nu} \overline{X}_\mu \overline{X}_\nu, \quad H^{(B)} = \frac{1}{2} \sum_{\mu,\nu} \overline{\delta}^{(B)}_{\mu,\nu} \overline{X}_\mu \overline{X}_\nu$$

where the $\overline{X}$'s now denote numbers, namely, multiples of the unit states of the redundancies $X$. According to a theorem on quadratic forms (reference 14), there then always exists a linear transformation:

$$\overline{X}_\mu = \sum_{\nu=1}^{n-3} c_{\nu;\mu} X_\nu \quad (\mu = 1, 2, \ldots, n - 3) \tag{74'}$$

by which $H^{(L)}$ and $H^{(B)}$ are transformed into expressions with purely quadratic terms:

$$H^{(L)} = \frac{1}{2} \sum_{\mu=1}^{m-3} d_\mu^{(L)} X_\mu^2 , \quad H^{(B)} = \frac{1}{2} \sum_{\mu=1}^{m-3} d_\mu^{(B)} X_\mu^2 \tag{75}$$

The coefficients $c_{\nu,\mu}$ of the transformation are the (m - 3) particular solutions of the homogeneous system of equations:

$$\sum_{\nu=1}^{m-3} \left(\bar{\delta}_{\lambda,\nu}^{(L)} - \rho_\mu \bar{\delta}_{\lambda,\nu}^{(B)}\right) c_{\nu,\mu} = 0 \quad (\lambda = 1, 2, \ldots, m - 3) \tag{76}$$

whose determinant, set equal to zero (an equation of the $(m - 3)^d$ degree), has the characteristic numbers $\rho_\mu$ $(\mu = 1, 2, \ldots, m - 3)$ as roots. If the transformation

$$X_\mu = \sum_{\nu=1}^{m-3} c_{\mu,\nu} \bar{X}_\nu$$

is now applied to the unit state of the force groups, the mixed displacement coefficients from stringers and sheet for the force groups X vanish, since for $\delta_{\mu,\nu}$, we have the transformation:

$$\delta_{\mu,\nu} = \sum_{p,q=1}^{m-3} \bar{\delta}_{p,q} \, c_{\mu,p} \, c_{\nu,q} \tag{77}$$

and for $H^{(L)}$ we have, after substituting from (74') and (77):

$$H^{(L)} = \frac{1}{2} \sum_{p,q} \bar{\delta}_{p,q}^{(L)} \left(\sum_\mu c_{\mu,p} X_\mu\right) \left(\sum_\nu c_{\nu,q} X_\nu\right) =$$

$$= \frac{1}{2} \sum_{\mu,\nu} X_\mu X_\nu \left[\sum_{p,q} \bar{\delta}_{p,q}^{(L)} c_{\mu,p} c_{\nu,q}\right]$$

$$= \frac{1}{2} \sum_{\mu,\nu} \delta_{\mu,\nu}^{(L)} X_\mu X_\nu = \frac{1}{2} \sum_\mu d_\mu^{(L)} X_\mu^2$$

so that,

$$\delta_{\mu,\nu}^{(L)} = d_\mu^{(L)} \text{ for } \mu = \nu, \quad \delta_{\mu,\nu}^{(L)} = 0 \text{ for } \mu \neq \nu$$

Similarly, for $\delta_{\mu,\nu}^{(B)}$.

The determination of orthogonal characteristic functions by neglecting one of the three essential stiffnesses with respect to their mutual coupling effects is thus reduced to the problem of the simultaneous transformation of two positive definite quadratic forms into expressions with only square terms. In the case of $n$ stringers, this requires the complete solution of a characteristic equation of the $(n - 3)^d$ degree and the computation of the corresponding particular solutions of the linear homogeneous system of equations (76). These algebraic operations for a finite number of variables correspond to the solution of the linear homogeneous integral equation in the work of Wagner and Simon.

Translation by S. Reiss,
National Advisory Committee
for Aeronautics.

REFERENCES

1. Schapitz, E., and Krümling, G.: Load Tests on a Stiffened Circular Cylindrical Shell. T.M. No. 864, N.A.C.A., 1938.

2. Ebner, H.: The Strength of Shell Bodies - Theory and Practice. T.M. No. 838, N.A.C.A., 1937.

3. Wagner, H., and Simon, H.: Über die Krafteinleitung in dünnwandige Zylinderschalen. Luftfahrtforschung, vol. 13, 1936, pp. 293-308.

4. Heck, O. S.: Über die Berechnung versteifter Scheiben und Schalen. Jahrbuch der deutschen Luftf.-Forschg, 1937, I, pp. 442-451.

5. Ebner, H., and Köller, H.: Über die Einleitung von Längskräften in versteifte Zylinderschalen. Jahrbuch der deutschen Luftf.-Forschg., 1937, I, pp. 464-473.

6. Ebner, H.: Zur Berechnung räumlicher Fachwerke im Flugzeugbau. DVL Report 138; Luftfahrtforschung, vol. 5, 1929, pp. 31-74; and DVL Yearbook, 1929, pp. 371-414. (Continued on p. 61.)

6 (cont.)

Ebner, H.: Die Berechnung regelmässiger vielfach statisch unbestimmter Raumfachwerke mit Hilfe von Differenzengleichungen. DVL Report 235; DVL Yearbook, 1931, pp. 246-288.

7. Ebner, H.: Torsional Stresses in Box Beams with Cross Sections Partially Restrained against Warping. T.M. No. 744, N.A.C.A., 1934.

8. Southwell, R. V., and Owen, J. B. B.: On the Calculation of Stresses in Braced Frameworks. R. & M. No. 1573, British A.R.C., 1935.

9. Hertwig, Müller-Breslau, and Pirlet: Eisenbau, vols. 1, 7, and 8.

Thalau-Teichmann: Aufgaben aus der Flugzeugstatik, Berlin, 1933.

Bayer, K.: Die Statik im Eisenbetonbau, 2. Aufl., Berlin, 1933, p. 230 ff.

10. Mises u. Pollaczek-Geiringer: Praktische Verfahren der Gleichungsauflösung. Z.f.a.M.M., vol. 9, 1929, pp. 58-77.

11. Funk, P.: Die linearen Differenzengleichungen und ihre Anwendung in der Theorie der Baukonstruktionen. Berlin, 1920.

Bleich, F., and Melan, E.: Die gewöhnlichen und partiellen Differenzengleichungen der Baustatik. Berlin, 1927.

12. Pohl, K.: Berechnung der Ringversteifungen dünnwandiger Hohlzylinder. Stahlbau (supplement to Bautechnik), 1931, pp. 157-163.

Lundquist, Eugene E., and Burke, Walter F.: General Equations for the Stress Analysis of Rings. T.R. No. 509, N.A.C.A., 1934.

Drescher, K., and Gropler, H.: Stresses in Reinforcing Rings Due to Axial Forces in Cylindrical and Conical Stressed Skin. T.M. No. 847, N.A.C.A., 1938.

13. Grüning, M.: Die Statik des ebenen Tragwerks. Berlin, 1935, pp. 157-160.

14. Courant-Hilbert: Methoden der mathematischen Physik, 2. Aufl., Berlin, 1931, pp. 32, 33.

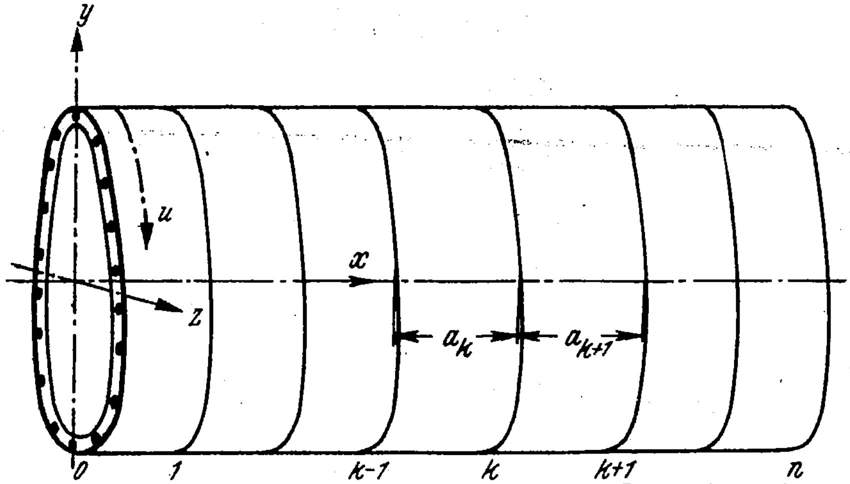
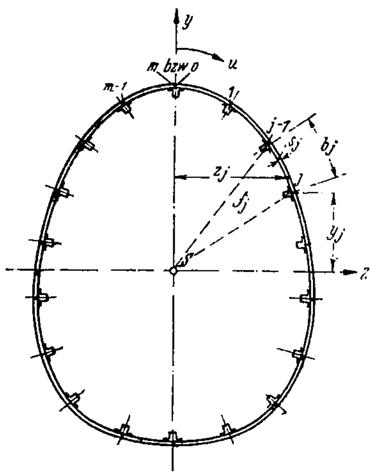
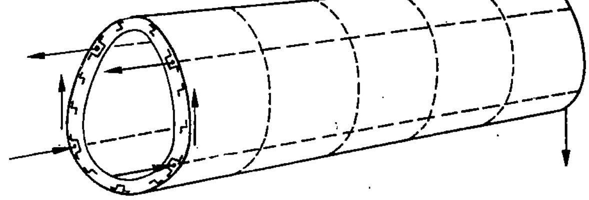



Figure 2.- Sketch of shell indicating notation used.




a, Concentrating bending forces.



b, Concentrated distorting forces.

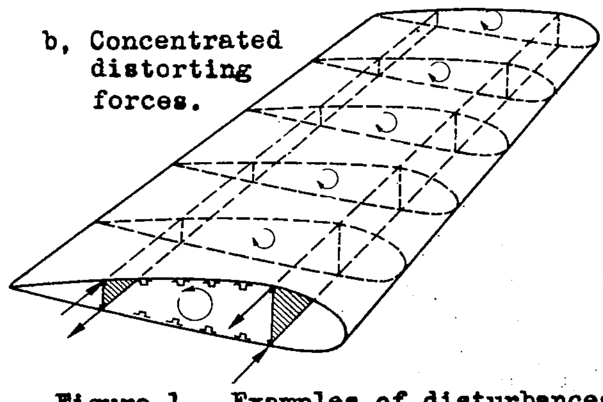
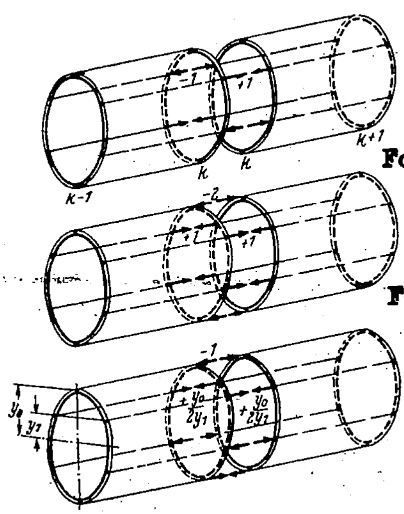


Figure 1.- Examples of disturbances from the elementary stress states.




Force group $X_{1,k}$

Force group $X_{2,k}$

Force group $X_{3,k}$

Figure 3.- Force groups for six-stringer shell.

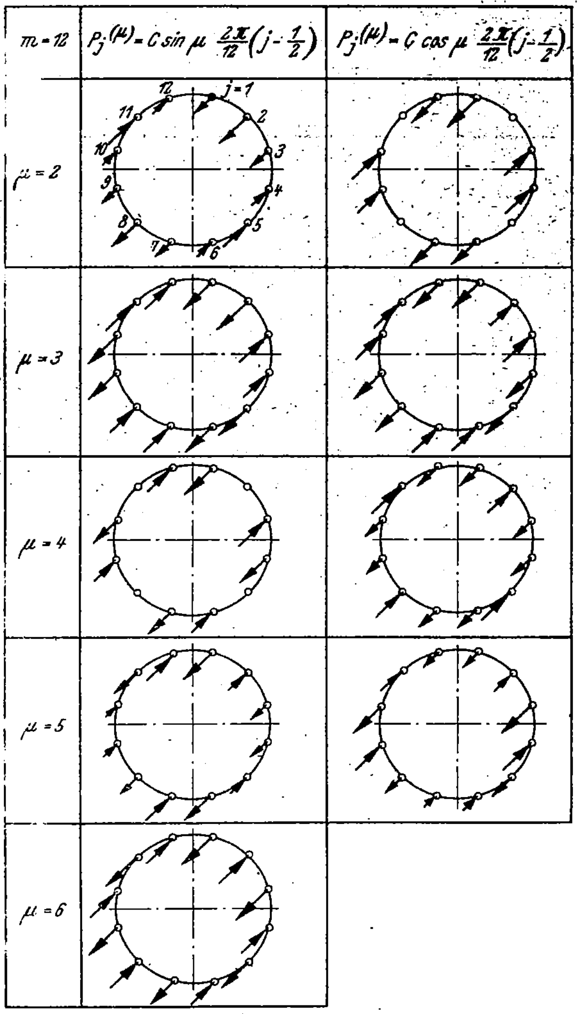
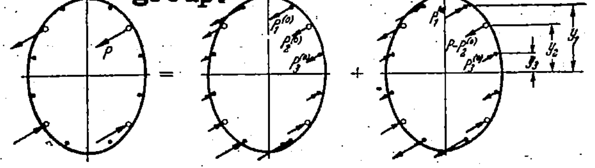



Figure 4.- Force groups for 12-stringer shell with cyclic symmetry.

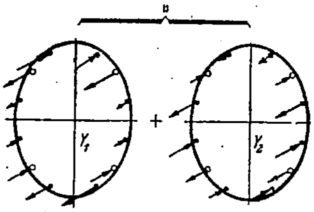
Bending force group = linearly distributed group + characteristic force group.



$$P_1^{(0)} = P \frac{F_1 y_1 y_2}{F_1 y_1^2 + F_2 y_2^2 + F_3 y_3^2}$$

$$P_2^{(0)} = P \frac{F_2 y_2^2}{F_1 y_1^2 + F_2 y_2^2 + F_3 y_3^2}$$

$$P_3^{(0)} = P \frac{F_2 y_1 y_3}{F_1 y_1^2 + F_2 y_2^2 + F_3 y_3^2}$$



Simply-symmetrical redundancies

Figure 5.- Example of the splitting up of applied longitudinal concentrated forces.

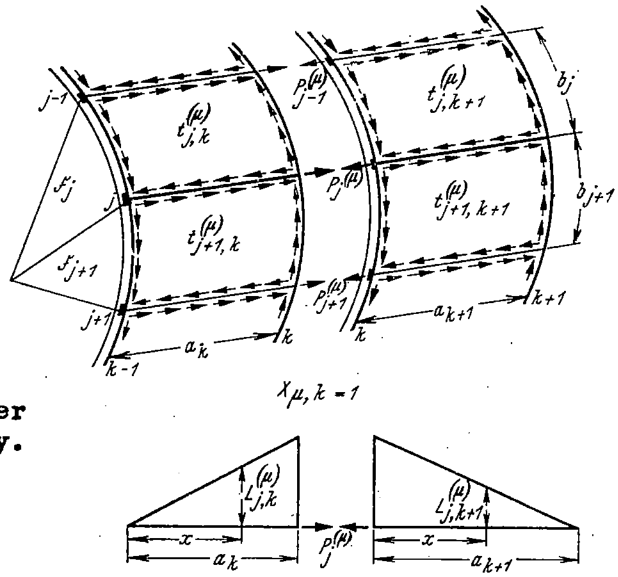


Figure 6.- Stress distribution in principal bay system due to $X_{u,k} = 1$.

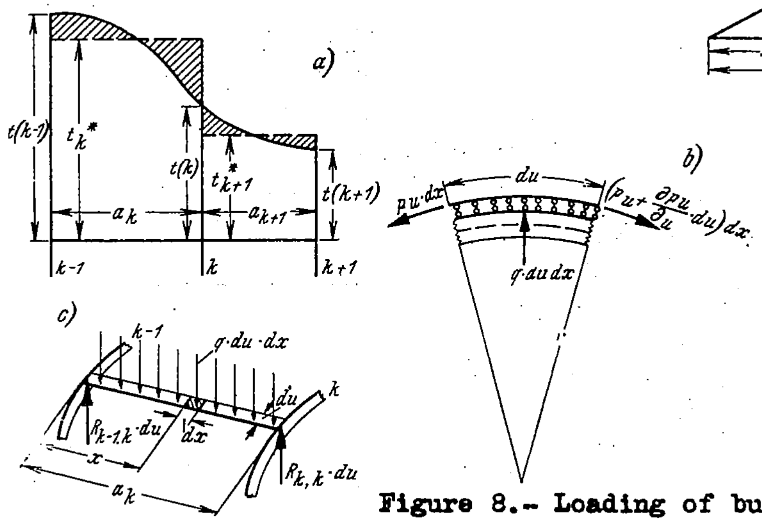


Figure 8.- Loading of bulkhead by radial forces

| μ | k | Group $X_{1,k}$ | | | | | | Group $X_{2,k}$ | | | | | | $\delta^{(\mu)}_{k,0}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 4 | 5 | 6 | 1 | 2 | 3 | 4 | 5 | 6 | |
| 1 | 1 | — | — | — | | | | — | — | — | | | | — |
| | 2 | — | — | — | — | | | — | — | — | — | | | — |
| | 3 | — | — | — | — | — | | — | — | — | — | — | | — |
| | 4 | | — | — | — | — | — | | — | — | — | — | — | — |
| | 5 | | | — | — | — | — | | | — | — | — | — | — |
| | 6 | | | | — | — | — | | | | — | — | — | — |
| 2 | 1 | — | — | — | | | | — | — | — | | | | — |
| | 2 | — | — | — | — | | | — | — | — | — | | | — |
| | 3 | — | — | — | — | — | | — | — | — | — | — | | — |
| | 4 | | — | — | — | — | — | | — | — | — | — | — | — |
| | 5 | | | — | — | — | — | | | — | — | — | — | — |
| | 6 | | | | — | — | — | | | | — | — | — | — |

Figure 7.- Scheme for elasticity equations for n = 6 and m = 5.

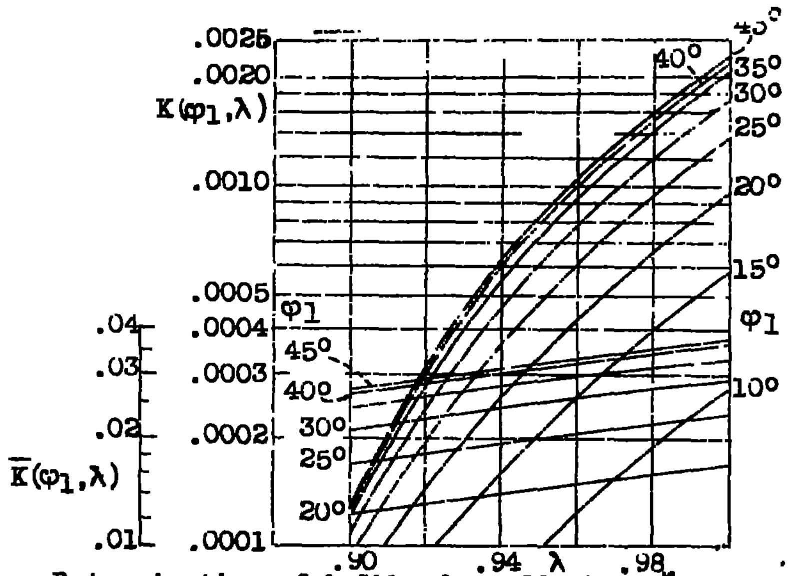



Figure 11.- Determination of bulkhead coefficient $\kappa_{S,k}$ for the applied convexing force group $X_k = 1$.

$$\kappa_{S,k} = \sqrt{\frac{\lambda_{S,k}}{\lambda_L}}\; K\left(\varphi_1, \sqrt{\lambda_L\, \lambda_{S,k}}\right) + \left(\frac{i_{S,k}}{r}\right)^2 \frac{\lambda_{S,k}}{\lambda_L}\; \bar{K}\left(\varphi_1, \lambda_L\right)$$

for tangential loading (bulkhead)

$$\kappa'_{S,k} = \left[\lambda^2_{S,k} + \left(\frac{i_{S,k}}{r}\right)^2\right] \frac{\lambda_{S,k}}{\sqrt{\lambda_L}}\; K\left(\varphi_1, \sqrt{\lambda_L}\right)$$

for radial bulkhead loading

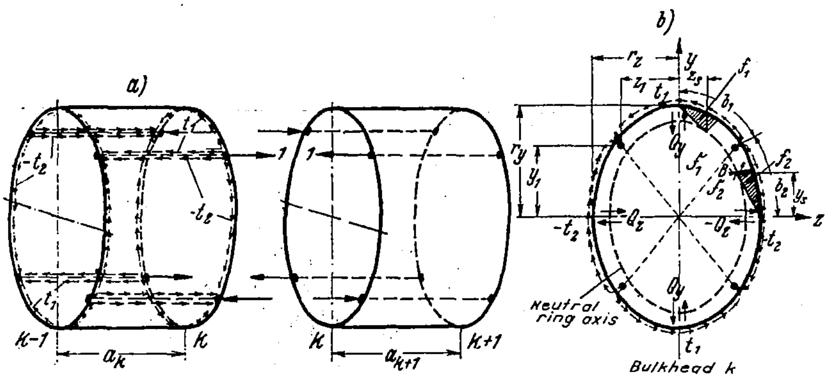
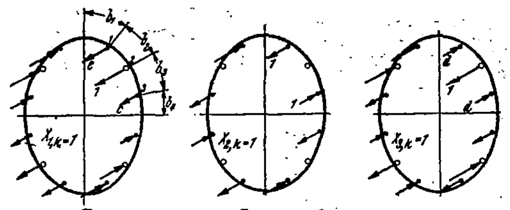
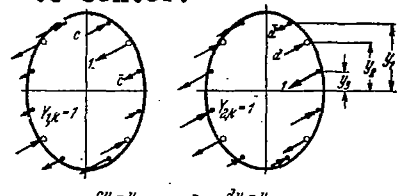



Figure 9.- Stress condition due to applied convexing force group $X_k = 1$.



a, Symmetrical with respect to center.



$\bar{c} = \frac{c y_1 - y_2}{y_3}$ ; $\bar{d} = \frac{d y_2 - y_3}{y_1}$

b, Simply symmetrical

Figure 13.- Characteristic force groups for 12-stringer shell with doubly-symmetrical cross section.

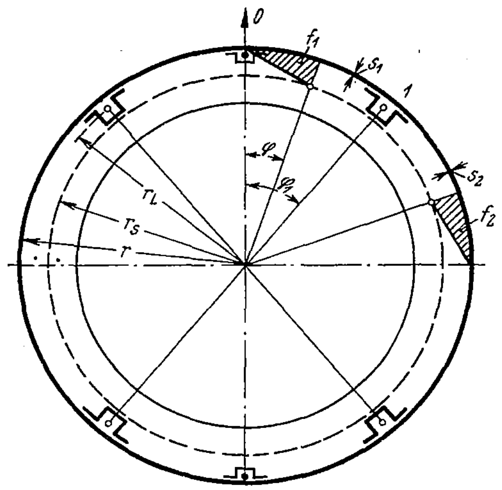


Figure 10.- Section of a circular cylindrical shell.

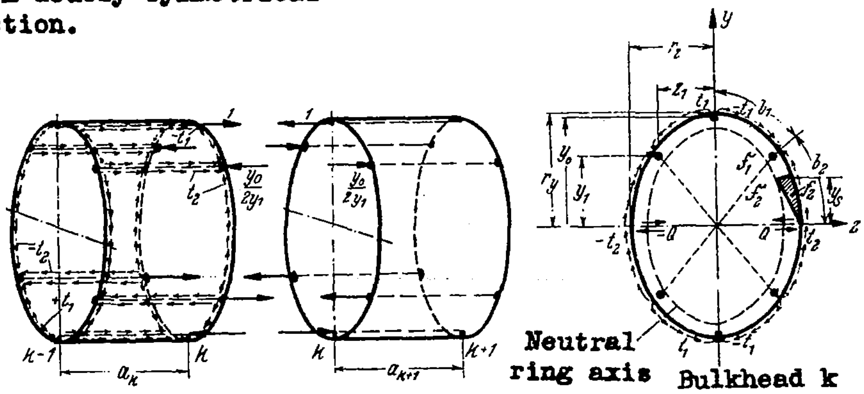



Figure 12.- Stress condition due to simply-symmetrical characteristic force group $Y_k = 1$.

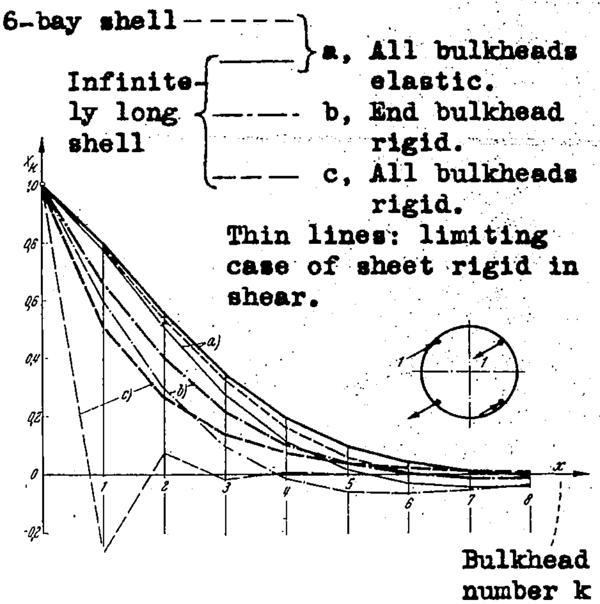



Figure 14.- Falling off of applied convexing force group $X_0 = 1$ over shell length.

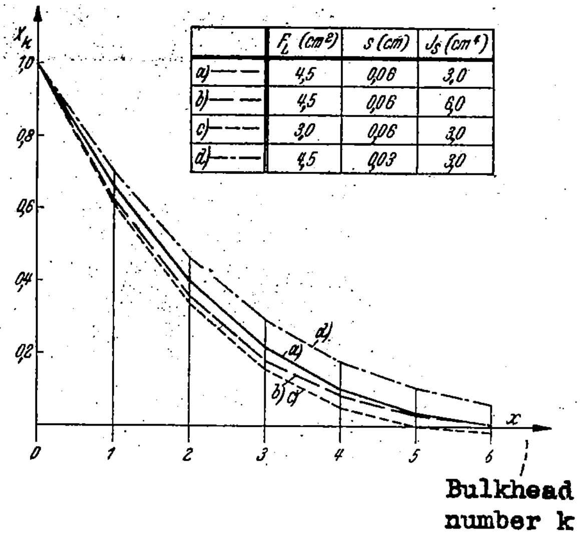
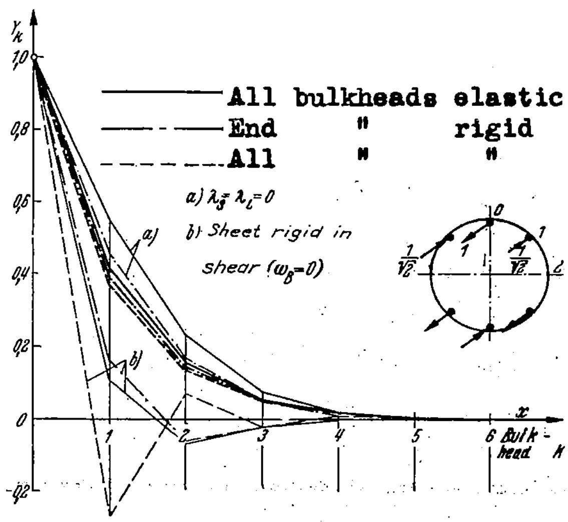



Figure 15.- Effect of the various stiffnesses on the distribution of the applied force group $X_0 = 1$.



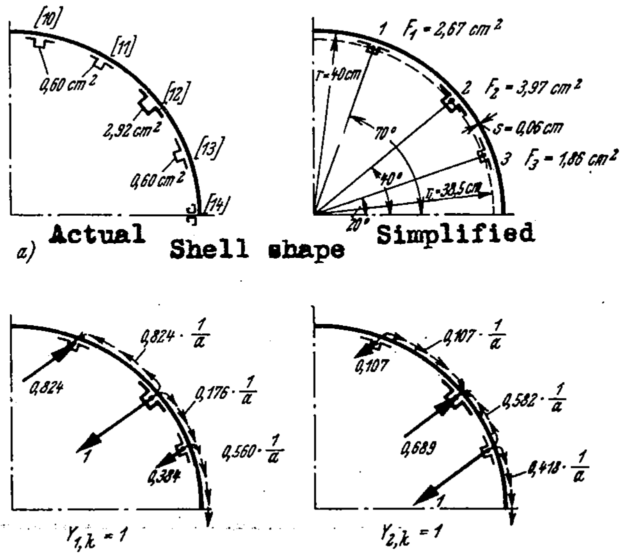




Figure 16.- Falling off of simply-symmetrical characteristic force group $Y_0 = 1$ over shell length.

b, Effective redundant force groups

Figure 17.- Bending force groups applied to the test shell.

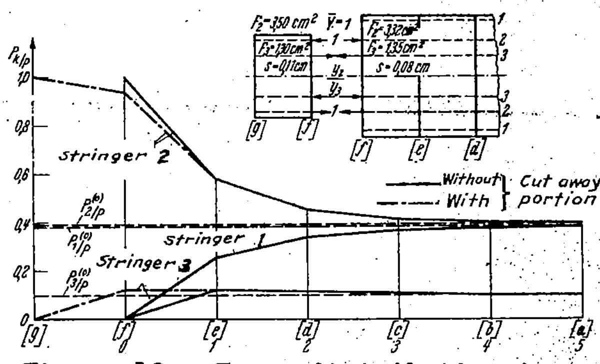



Figure 18.- Force distribution in the simplified shell due to the bending force group

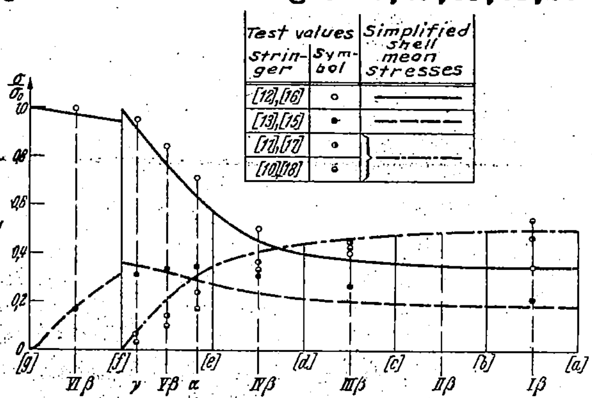
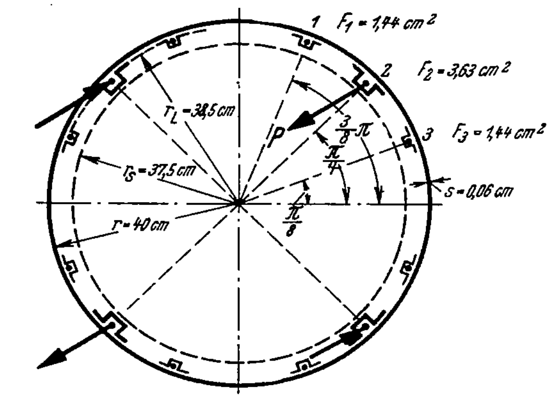
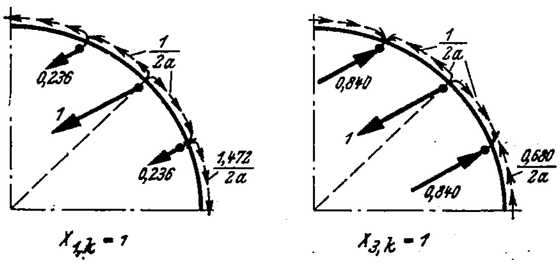



Figure 19.- Comparison of computed and measured stringer stresses due to the bending force group.




a, Simplified shell shape




b, Effective force groups

Figure 20.- Convexing force group applied to the test shell.

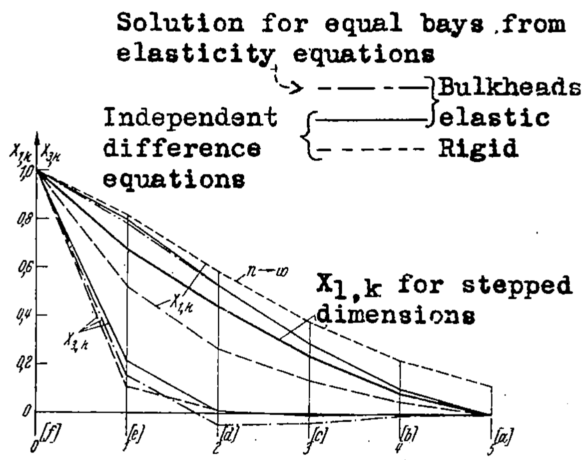



Figure 21.- Distribution of the antisymmetrical characteristic force groups $X_{1,k} = 1$ and $X_{3,k} = 1$ in the simplified shell.

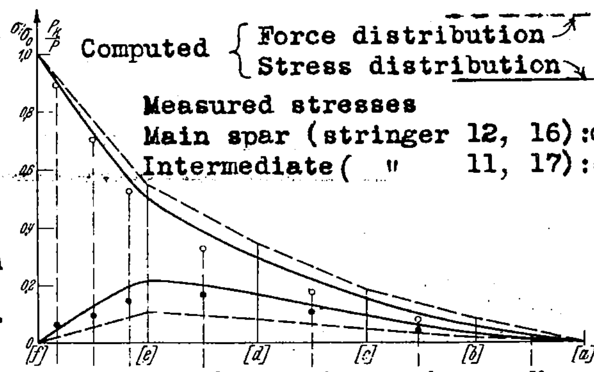



Figure 22.- Force distribution and comparison of computed and measured stringer stresses due to convexing force group.